Tensors, Einstein Summation Notation

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1 - Tensors

- A generalization of scalars (rank-0 tensors) and vectors (rank-1 tensors)
- A linear map from the space of rank-n tensors to the space of rank-m tensors is represented as a rank (m+n) tensor. For example: a linear operator that maps vectors to vectors is represented by a rank-2 tensor (a matrix).
- Coordinates. Scalars, vectors, and tensors exist independently of coordinates. However, given a choice of coordinate axes, they can be represented in those coordinates.

Let $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$

Let A be a linear map from vectors to vectors, and suppose that $\vec{u} = A(\vec{v})$, i.e.,

$$u_1 = a_{11}v_1 + a_{12}v_2 + a_{13}v_3 u_2 = a_{21}v_1 + a_{22}v_2 + a_{23}v_3 u_3 = a_{31}v_1 + a_{32}v_2 + a_{33}v_3$$

where, a_{ij} $(i, j \, \epsilon \, \{1, 2, 3\})$ encodes any linear map between vectors in $\Re^3$ to vectors in $\Re^3$

Therefore given a choice of coordinate frame,

A can be represented by the matrix

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

N.B. The matrix a_{ij} depends on the coordinate frame, but A is independent of the coordinate frame.

2 - Einstein Summation Notation

- Think of k-dimensional arrays of numbers:

$$\begin{array}{cc} k = 0 & [a] \\ k = 1 & [a_1, a_2, \ldots, a_n] \\ k = 2 & \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{23} \\ \vdots & & \ddots & \\ a_{m1} & & & a_{mn} \end{pmatrix} \\ \vdots & \ldots \end{array}$$

- Consider summations, e.g., $\sum_i a_i b_i$ and rewrite as: $a_i b_i$, omitting the summation sign, easy!
- Rule: if an index appears twice on the *same* side of the equal sign, sum over range of index. Examples:

$$
\begin{aligned}
\vec{a} \cdot \vec{b} = & \quad a_i b_i \\
||\vec{a}||^2 = & \quad a_i a_i \\
\vec{a} = \vec{b} \Rightarrow & \quad a_i = b_i (\text{no summation}) \\
\vec{u} = A\vec{v} \Rightarrow & \quad u_i = a_{ij} b_{ij} (\text{sum over j, but not i}) \\
A = BC \Rightarrow & \quad a_{ik} = b_{ij} c_{jk} (\text{matrix} - \text{matrix multiplication})
\end{aligned}
$$

Two important tensors:

- The Kronecker Delta: δ :

$$\delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

  The matrix representation is the familiar I matrix.

  E.g., Trace of a rank-2 tensor: $Tr(A) = a_{ij}\delta_{ij}$

- Permutation Symbol/Permutation Tensor

$$\epsilon_{ijk} = \begin{cases} 1 & \text{the arguments are an even permutation of } 1,2,3 \\ -1 & \text{the arguments are an odd permutation of } 1,2,3 \\ 0 & \text{two or more arguments are the same} \end{cases}$$

  Example: $\epsilon_{123} = 1$, $\epsilon_{132} = -1$, $\epsilon_{122} = 0$.

  Note: $\epsilon_{ijk} = -\epsilon_{ikj}$

  The permutation tensor can be used to define the vector cross product.

Revision	**Date**	**Description**
1.0	January 1, 2014	Creation
1.1	February 12, 2023	Reformatting