Derivation of discrete bending forces and their gradients

Rasmus Tamstorf

Walt Disney Animation Studios

July 10, 2013

Abstract.   Previous papers have introduced discrete bending energies for thin shell and cloth simulations. In this report the corresponding bending forces and their gradients are derived. The derivation only requires basic trigonometry and repeated use of the chain rule for differentiation. By observing all the symmetries in the derivation it is shown that the evaluation of the bending forces and gradients can be sped up $3\times$ compared to a naive implementation.

1.   Notation

$\mathbf{x}$	Position in deformed configuration
$\bar{\mathbf{x}}$	Position in undeformed configuration
W	Energy per element
A	Triangle area
l	Edge length
s	Edge strain
θ	Bend angle
$\mathbf{e}$	Edge vector
$\mathbf{n}$	Normal vector
$\mathbf{t}$	In-plane edge normal
$\mathbf{f}$	Force vector
H	Energy Hessian or mean curvature (average of principal curvatures)
H_Σ	Sum of principal curvatures

a	Scalar	Lower case, latin
$\mathbf{x}$	Vector	Lower case, bold, latin
$\hat{\mathbf{x}}$	Normalized vector	Hat over vector
$\bar{\mathbf{x}}$	Undeformed vector (material space)	Bar over vector
$\dot{\mathbf{x}}$	Time derivative of vector	Dot over vector
M	Matrix (second order tensor)	Upper case, latin

2. Discrete shells

The bending energy for a smooth thin shell can be written as a function of the change in mean curvature squared. In [Grinspun et al. 03] it is shown how this can be discretized and expressed in terms of the dihedral angle between adjacent triangles in a mesh. We shall use the resulting expression as the foundation for our derivations here. In particular, we use the following bending energy :

$$E_b(\mathbf{x}) = k_b \frac{3\|\bar{\mathbf{e}}_0\|^2}{\bar{A}}(\theta - \bar{\theta})^2 \tag{1}$$

In this equation k_b is the bend stiffness, $\mathbf{e}_0$ is the edge along a hinge, θ is the *bend angle* for the hinge, and $\bar{\theta}$ denotes the same angle in the undeformed configuration. The bend angle is the angle between the normals of the two triangles incident to the hinge which is the same as the complement of the *dihedral* angle for the hinge. The bend angle is zero for a flat material.

In the following sections we let $\mathbf{x} = (\mathbf{x}_0^T, \mathbf{x}_1^T, \mathbf{x}_2^T, \mathbf{x}_3^T)^T$ be the four vertices of the two triangles stacked together into a single vector. This is all illustrated in figure 1 and 2 which are similar to those in the appendix of [Wardetzky et al. 07]. One of the modifications that should be noted is that we follow the original notation from [Bridson et al. 03] and [Grinspun et al. 03] and use θ to denote the bend angle whereas [Wardetzky et al. 07] uses θ to denote the dihedral angle.
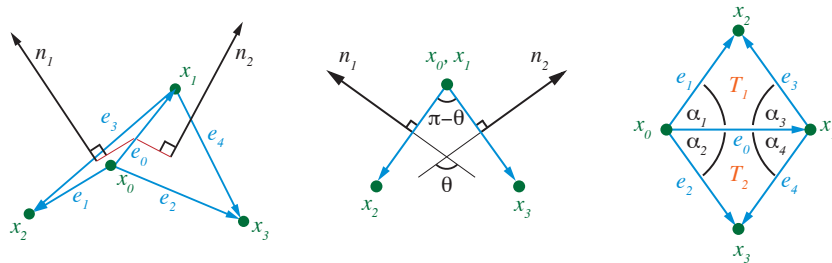

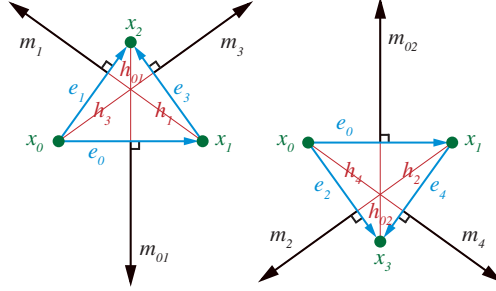**Figure 1.** *Vertices, edges, normals and angles around the edge shared by two triangles.*

Figure 2. *Inplane edge normals and the associated altitudes from one edge to the opposing vertex.*

The derivation of the above energy contains a couple of subtle details which we shall briefly clarify. First, it should be noted that mean curvature is sometimes defined as the *sum* of the principal curvatures and sometimes as the *average* of the principal curvatures. We will use H_Σ to denote the sum and simply H to denote the average.

With this notation, we note that the appendix of [Grinspun et al. 03] refers to a result by [Cohen-Steiner and Morvan 03] to assert that :

$$\int_{\mathcal{T}} \bar{H}_\Sigma d\bar{A} = \bar{\theta} \|\bar{\mathbf{e}}_0\| \tag{2}$$

where and $\mathcal{T}$ is the diamond region associated with a hinge. By "a similar argument" they state that :

$$\int_{\mathcal{T}} \Delta H_\Sigma d\bar{A} = (\theta - \bar{\theta}) \|\bar{\mathbf{e}}_0\|$$

What is important to note here, is that [Cohen-Steiner and Morvan 03] uses H_Σ as their mean curvature. When averaged across the two triangles this gives an approximation for the pointwise difference in H_Σ. If $\bar{A} = \bar{A}_1 + \bar{A}_2$ is the area of the two triangles incident to the hinge in the rest configuration then the area associated with the hinge is a third of that, so the estimate for the pointwise difference becomes :

$$\Delta H_\Sigma \approx \frac{3}{\bar{A}} \int \Delta H_\Sigma d\bar{A} = 3(\theta - \bar{\theta}) \|\bar{\mathbf{e}}_0\| / \bar{A}$$

In [Grinspun et al. 03] it is in effect argued that for sufficiently small triangles, the mean curvature difference can be assumed to be constant across the triangle. As a consequence it follows that the average of the square of the

difference is the same as the square of the average difference. I.e.

$$\left(\int \Delta H_\Sigma dA\right)^2 \approx \int (\Delta H_\Sigma)^2 dA$$

This is obviously only true for infinitely small triangles (i.e. pointwise), while it becomes an approximation for finite size triangles. However, assuming that this approximation is valid, we get the following pointwise estimate :

$$(\Delta H_\Sigma)^2 = 9(\theta - \bar{\theta})^2 \|\bar{\mathbf{e}}_0\|^2 / \bar{A}^2. \tag{3}$$

When integrated across the support region of the hinge (of area $\bar{A}/3$), this gives a contribution to the energy of

$$\int (\Delta H_\Sigma)^2 d\bar{A} = 3(\theta - \bar{\theta})^2 \|\bar{\mathbf{e}}_0\|^2 / \bar{A}. \tag{4}$$

In the main body of [Grinspun et al. 03] this expression is used as a discretization of

$$\int (\Delta H_\Sigma)^2 d\bar{A} = \int_{\bar{\Omega}} (H_\Sigma \circ \phi - \bar{H}_\Sigma)^2 d\bar{A} = \int_{\bar{\Omega}} 4(H \circ \phi - \bar{H})^2 d\bar{A}$$

where ϕ is the deformation function. Note that although the paper only ever uses H to denote mean curvature, it is using it to refer to the *average* of the principal curvatures in the main body, but the *sum* of the principal curvatures in the appendix.

As a simple test for an implementation, one might consider computing the integrated mean curvature over a sphere. For a sphere we have $H = -1/r$ where r is the radius and thus $H_\Sigma = -2/r$. Assuming a flat rest configuration it follows that $(\Delta H_\Sigma)^2 = 4/r^2$, and since the area of a sphere is $4\pi r^2$ we get :

$$\int (\Delta H_\Sigma)^2 d\bar{A} = 16\pi$$

Since the resulting energy is constant (and independent of radius) it also follows that there should be no net force. Both of these facts can be checked in an implementation.

We may choose to use different formulations for the discrete curvature by modifying Equation (2). In that case, the discrete energy becomes :

$$E_b(\mathbf{x}) = k_b \frac{3\|\bar{\mathbf{e}}_0\|^2}{\bar{A}} (\varphi(\theta) - \varphi(\bar{\theta}))^2 \tag{5}$$

where $\varphi(\cdot)$ is a discrete curvature measure corresponding to H_Σ. One such measure that we shall consider is :

$$\varphi(\theta) = 2\tan\left(\frac{\theta}{2}\right) \tag{6}$$

We generally don't allow material to go through itself, so we expect that $\theta \in [-\pi; \pi[$ and similarly for $\bar{\theta}$. As a consequence $\theta - \bar{\theta}$ need *not* be in $[-\pi; \pi[$, but may in fact be in any subinterval of $[-2\pi; 2\pi[$ of length 2π. It is therefore important to note that the energy is based on $\varphi(\theta) - \varphi(\bar{\theta})$ rather than $\varphi(\theta - \bar{\theta})$ (this is consistent with [Gingold et al. 04]). If we were to use $\varphi(\theta - \bar{\theta})$ then the energy would contain discontinuities when using the discrete curvature measure given in Equation (6). Several papers *do* use $\varphi(\theta - \bar{\theta})$, and for small (infinitesimal) bend-angles all of these methods are equivalent (the curvature approaches zero as the bend-angles approaches zero). However, for large (finite) bend-angles the distinction becomes important.

In order to be able to determine which way a shell is bending, the bend-angle, θ, has to be computed as a signed quantity. If the rest-angle is zero (i.e., when we consider a thin plate), then the sign doesn't matter because all the discrete mean curvature estimates are odd functions such that $H(-\theta) = -H(\theta)$. Since the bending energy is a function of the square of the mean curvature, the actual sign therefore doesn't matter in this case. However, for thin shells where the rest-angle is non-zero, the sign must be computed in a consistent manner. We use the same (arbitrary) convention as in [Bridson et al. 03] and let θ have the same sign as $(\mathbf{n}_1 \times \mathbf{n}_2) \cdot \mathbf{e}_0 = \det[\mathbf{n}_1, \mathbf{n}_2, \mathbf{e}_0]$. This means that θ is positive if the two normals are pointing away from each other, i.e., if the shape is convex, which is also consistent with [Cohen-Steiner and Morvan 03].
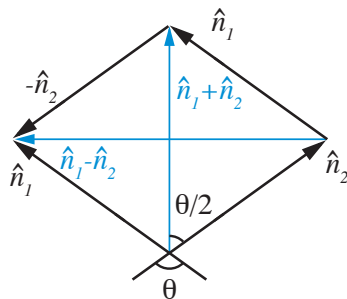

Figure 3. *Simple construction to compute the trigonometric functions for $\frac{\theta}{2}$ based on one of the right angled triangles. Note that each of the hypotenuses have unit length.*

To compute the various trigonometric functions for $\frac{\theta}{2}$ for $\theta \in [0; \pi]$ we see

from figure 3 that :

$$\sin\left(\tfrac{\theta}{2}\right) = \frac{\|\hat{\mathbf{n}}_1 - \hat{\mathbf{n}}_2\|}{2} \tag{7}$$

$$\cos\left(\tfrac{\theta}{2}\right) = \frac{\|\hat{\mathbf{n}}_1 + \hat{\mathbf{n}}_2\|}{2} \tag{8}$$

$$\tan\left(\tfrac{\theta}{2}\right) = \frac{\|\hat{\mathbf{n}}_1 - \hat{\mathbf{n}}_2\|}{\|\hat{\mathbf{n}}_1 + \hat{\mathbf{n}}_2\|} \tag{9}$$

For values of θ outside the specified interval the usual identities for trigonometric functions can be used, which means that the expressions for sin and tan simply have to be multiplied by the sign of θ. We note that for $\theta = 0$ the sign function is not well defined, but the numerator for both $\sin\left(\tfrac{\theta}{2}\right)$ and $\tan\left(\tfrac{\theta}{2}\right)$ is zero. Hence the sign function can be defined arbitrarily in this situation as long as it is finite.

In order to determine the proper value for the bending stiffness, k_b, we note that while the expressions here are used for general shells, the model must be consistent with a plate model given a flat rest state. According to [Audoly and Pomeau 10, Eq. 6.95b], the bending energy for a plate undergoing small deformations is :

$$E_b = \frac{D}{2} \int (\Delta w)^2 dA$$

where D is the *flexural rigidity*, Δ is used to denote the Laplace operator and w is the height field of displacements across the plate. The Laplacian of a height field, w, gives the first order approximation of the sum of the principal curvatures, so effectively this states that :

$$E_b \approx \frac{D}{2} \int (H_\Sigma)^2 dA$$

By writing the flexural rigidity in terms of Young's modulus, Y, Poisson's ratio, ν, and the thickness of the plate or shell, h, it follows that :

$$k_b = \frac{D}{2} = \frac{Yh^3}{24(1-\nu^2)}.$$

3.   Vector gradients

Let $\mathbf{x} \in \mathbb{R}^n$ be a point in the configuration space. For the purposes of the following, n is typically 12 as it contains the coordinates of the 4 vertices in $\mathbb{R}^3$ which make up the stencil of a hinge. Also, let $\mathbf{u}(\mathbf{x})$ and $\mathbf{v}(\mathbf{x}) \in \mathbb{R}^m$ be vector valued functions which depend on the configuration. Typically m will be 3

as we compute normal and edge vectors as functions of the underlying configuration. Vectors are assumed to be column vectors such that the gradient, $\nabla = \left[\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right]$, becomes a row vector.

In the following we will build a library of gradients of functions of vectors along with some intuition about how to interpret them. First consider the gradient of $\|\mathbf{x}\|^2$ which is a scalar valued function of $\mathbf{x}$:

$$\nabla\|\mathbf{x}\|^2 = \nabla \sum_{i=0}^{n} x_i^2 = 2\mathbf{x}^T$$

The gradient of the norm itself then follows by using the chain-rule for a scalar function

$$\nabla\|\mathbf{x}\| = \nabla\sqrt{\|\mathbf{x}\|^2} = \frac{1}{2\sqrt{\|\mathbf{x}\|^2}}\nabla\|\mathbf{x}\|^2 = \frac{\mathbf{x}^T}{\|\mathbf{x}\|} = \hat{\mathbf{x}}^T$$

Here we use a hat to denote the normalized vector. The interpretation of this is simply that the largest change in the norm of $\mathbf{x}$ is obtained by moving along the vector $\mathbf{x}$ itself.

Let us now consider the gradient of $\mathbf{u}(\mathbf{x})$. Since $\mathbf{u}$ is a vector, it follows that $\nabla\mathbf{u}$ is a matrix where each row is the gradient of the corresponding component of $\mathbf{u}$. This is also known as the Jacobian, but we will not make the distinction and simply refer to it as the gradient of $\mathbf{u}$. Using this notation, the chain-rule for a function, $f(\mathbf{u}(\mathbf{x}))$, states that

$$\nabla_x f = \nabla_u f \nabla_x \mathbf{u}$$

Similarly for a scalar-valued function, f, the multivariate product rule for $f(\mathbf{x})\mathbf{u}(\mathbf{x})$ can be written as

$$\nabla(f\mathbf{u}) = f\nabla\mathbf{u} + \mathbf{u}\nabla f$$

It should be noted that the order in each of the two products is important since the matrix product is not commutative. By applying the chain-rule we can now easily extend the result above :

$$\nabla\|\mathbf{u}\| = \hat{\mathbf{u}}^T \nabla\mathbf{u}$$

and it follows that :

$$\nabla\left(\frac{1}{\|\mathbf{u}\|}\right) = \frac{-1}{\|\mathbf{u}\|^2}\nabla\|\mathbf{u}\| = -\frac{\hat{\mathbf{u}}^T \nabla\mathbf{u}}{\|\mathbf{u}\|^2}$$

Based on the product-rule we also see that

$$\nabla\hat{\mathbf{u}} = \nabla\left(\frac{\mathbf{u}}{\|\mathbf{u}\|}\right) = \mathbf{u}\nabla\left(\frac{1}{\|\mathbf{u}\|}\right) + \frac{1}{\|\mathbf{u}\|}\nabla\mathbf{u} = -\mathbf{u}\frac{\hat{\mathbf{u}}^T \nabla\mathbf{u}}{\|\mathbf{u}\|^2} + \frac{\nabla\mathbf{u}}{\|\mathbf{u}\|}$$

After simplification this gives

$$\nabla\hat{\mathbf{u}} = (I - \hat{\mathbf{u}}\hat{\mathbf{u}}^T)\frac{\nabla\mathbf{u}}{\|\mathbf{u}\|}$$

The factor $(I - \hat{\mathbf{u}}\hat{\mathbf{u}}^T)$ is a projection matrix, which projects away all components along $\hat{\mathbf{u}}$. To see this, notice that $(I - \hat{\mathbf{u}}\hat{\mathbf{u}}^T)\hat{\mathbf{u}} = \hat{\mathbf{u}} - \hat{\mathbf{u}}\|\hat{\mathbf{u}}\|^2 = \hat{\mathbf{u}} - \hat{\mathbf{u}} = 0$. Intuitively this is not surprising since any change along $\mathbf{u}$ will not affect $\hat{\mathbf{u}}$. The remaining change (which is orthogonal to $\mathbf{u}$) will affect $\hat{\mathbf{u}}$ more or less depending on how long $\mathbf{u}$ is. A small change at the end of a long vector will only turn $\hat{\mathbf{u}}$ a little, while the same change at the end of a very short vector will have a much bigger impact. Hence, the division by $\|\mathbf{u}\|$ also makes intuitive sense.

For the dot product of two vectors we have

$$\mathbf{u}(\mathbf{x}) \cdot \mathbf{v}(\mathbf{x}) = \sum_{j=1}^{m} u_j(\mathbf{x})v_j(\mathbf{x})$$

The i'th component of the gradient therefore becomes

$$(\nabla(\mathbf{u} \cdot \mathbf{v}))_i = \sum_{j=1}^{m} u_j\frac{\partial v_j}{\partial x_i} + v_j\frac{\partial u_j}{\partial x_i} = \mathbf{u}^T(\nabla\mathbf{v})_i + \mathbf{v}^T(\nabla\mathbf{u})_i$$

Putting all the components together we get

$$\nabla(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u}^T\nabla\mathbf{v} + \mathbf{v}^T\nabla\mathbf{u}$$

We can interpret this result geometrically based on the fact that $\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\|\cos\theta$ where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$. A change in $\mathbf{u} \cdot \mathbf{v}$ can arise in two ways : By changing $\mathbf{u}$ and by changing $\mathbf{v}$. This corresponds to the two terms (the first one accounts for change in $\mathbf{v}$, while the second one accounts for change in $\mathbf{u}$). If we let the variation $\delta\mathbf{u} = \nabla\mathbf{u}\delta\mathbf{x}$ be a small change in $\mathbf{u}$ due to a small change $\delta\mathbf{x}$ in $\mathbf{x}$, then we know that $\delta\mathbf{u}$ must be aligned with $\mathbf{u}$ to change $\|\mathbf{u}\|$, and it must be orthogonal to $\mathbf{u}$ in order to change θ. If the orthogonal component of $\delta\mathbf{u}$ is aligned with $\mathbf{v}$ then $\mathbf{v} \cdot \delta\mathbf{u}$ will be positive indicating that $\cos\theta$ increases which in turn means that θ is decreasing (since $\theta \in [0, \pi]$). If on the other hand $\delta\mathbf{u}$ and $\mathbf{v}$ are opposite, then θ is increasing. The situation is symmetric for $\delta\mathbf{v}$.

The cross product is a little more complicated, but can be computed componentwise just the same way as the dot product. The end result is

$$\nabla(\mathbf{u} \times \mathbf{v}) = \mathbf{u} \times \nabla\mathbf{v} - \mathbf{v} \times \nabla\mathbf{u}$$

The cross product between a vector and a matrix should here be understood as the cross product between the vector and each of the columns in the matrix.

Also in order to be able to write $\delta f = \nabla f \delta \mathbf{x}$ the expression $\nabla f \delta \mathbf{x}$ should be treated like a matrix product which is not commutative.

The next step is to consider the gradient of a normalized cross product. This will come up later as the gradient of a triangle normal. Let $\mathbf{n} = \mathbf{u} \times \mathbf{v}$. We then have :

$$
\begin{aligned}
\delta \hat{\mathbf{n}} &= \nabla \hat{\mathbf{n}} \delta \mathbf{x} \\
&= \left(I - \hat{\mathbf{n}} \hat{\mathbf{n}}^T \right) \frac{\nabla \mathbf{n}}{\|\mathbf{n}\|} \delta \mathbf{x} \\
&= \left(I - \hat{\mathbf{n}} \hat{\mathbf{n}}^T \right) \frac{\nabla (\mathbf{u} \times \mathbf{v})}{\|\mathbf{u} \times \mathbf{v}\|} \delta \mathbf{x} \\
&= \left(I - \hat{\mathbf{n}} \hat{\mathbf{n}}^T \right) \frac{\mathbf{u} \times \delta \mathbf{v} - \mathbf{v} \times \delta \mathbf{u}}{\|\mathbf{u} \times \mathbf{v}\|}
\end{aligned}
$$

Looking at each of the terms in the numerator of the fraction we see that if $\delta \mathbf{v}$ is parallel to $\mathbf{v}$ then $\mathbf{u} \times \delta \mathbf{v}$ will be parallel to $\mathbf{u} \times \mathbf{v} = \mathbf{n}$ which will be projected away. If $\delta \mathbf{v}$ is parallel to $\mathbf{u}$ then $\mathbf{u} \times \delta \mathbf{v}$ will be zero, so for any $\delta \mathbf{v}$ in the plane spanned by $\mathbf{u}$ and $\mathbf{v}$ (i.e., orthogonal to $\hat{\mathbf{n}}$) we get no contribution to $\delta \hat{\mathbf{n}}$. A similar argument holds for the second term. Consequently we see that $\delta \mathbf{u}$ and $\delta \mathbf{v}$ must be orthogonal to the plane spanned by $\mathbf{u}$ and $\mathbf{v}$ to affect $\delta \hat{\mathbf{n}}$; i.e., the change must be parallel to $\hat{\mathbf{n}}$ itself, which makes sense since any inplane deformation will not change the normal.

Formally, let $\delta \mathbf{u} = \delta \mathbf{u}_\| + \delta \mathbf{u}_\perp$ and $\delta \mathbf{v} = \delta \mathbf{v}_\| + \delta \mathbf{v}_\perp$ be the decomposition of the variation of $\mathbf{u}$ and $\mathbf{v}$ into components parallel with $\hat{\mathbf{n}}$ and orthogonal to $\hat{\mathbf{n}}$. The above expression then simplifies :

$$
\begin{aligned}
\delta \hat{\mathbf{n}} &= \left(I - \hat{\mathbf{n}} \hat{\mathbf{n}}^T \right) \frac{\mathbf{u} \times (\delta \mathbf{v}_\| + \delta \mathbf{v}_\perp) - \mathbf{v} \times (\delta \mathbf{u}_\| + \delta \mathbf{u}_\perp)}{\|\mathbf{n}\|} \\
&= \frac{\mathbf{u} \times \delta \mathbf{v}_\| - \mathbf{v} \times \delta \mathbf{u}_\|}{\|\mathbf{n}\|} \\
&= \frac{\mathbf{u} \times (\hat{\mathbf{n}} \hat{\mathbf{n}}^T \delta \mathbf{v}) - \mathbf{v} \times (\hat{\mathbf{n}} \hat{\mathbf{n}}^T \delta \mathbf{u})}{\|\mathbf{n}\|}
\end{aligned}
$$

If we write the cross products in terms of multiplication with the corresponding skew-symmetric matrix, then the associativity of the matrix product makes it clear that we can rewrite this as :

$$
\begin{aligned}
\delta \hat{\mathbf{n}} &= \frac{[\mathbf{u}]_\times \hat{\mathbf{n}} \hat{\mathbf{n}}^T \nabla \mathbf{v} - [\mathbf{v}]_\times \hat{\mathbf{n}} \hat{\mathbf{n}}^T \nabla \mathbf{u}}{\|\mathbf{n}\|} \delta \mathbf{x} \\
&= \frac{(\mathbf{u} \times \hat{\mathbf{n}}) \hat{\mathbf{n}}^T \nabla \mathbf{v} - (\mathbf{v} \times \hat{\mathbf{n}}) \hat{\mathbf{n}}^T \nabla \mathbf{u}}{\|\mathbf{n}\|} \delta \mathbf{x}
\end{aligned}
$$

It therefore follows that we have :

$$
\nabla \hat{\mathbf{n}} = \frac{(\mathbf{u} \times \hat{\mathbf{n}}) \hat{\mathbf{n}}^T \nabla \mathbf{v} - (\mathbf{v} \times \hat{\mathbf{n}}) \hat{\mathbf{n}}^T \nabla \mathbf{u}}{\|\mathbf{n}\|} \tag{10}
$$

4.   Gradient and Hessian of the bending energy

In order to compute the bending forces, we need to compute $-\nabla E_b$ and for an implicit solver we also need to compute the Jacobian of the bending forces (i.e., $\nabla((-\nabla E_b)^T) = -H(E_b)$).

   The gradient and the Hessian of the bending energy in Equation (5) can be expressed in terms of the gradient and the Hessian of the bend angle. Since all the quantities in front of $(\varphi(\theta) - \varphi(\bar{\theta}))^2$ refer to the rest configuration, they are effectively constants for purposes of differentiation, so we get

$$-\nabla E_b = -\nabla \left(k_b \frac{3\|\bar{\mathbf{e}}_0\|^2}{\bar{A}} (\varphi - \bar{\varphi})^2 \right) = k(\varphi - \bar{\varphi})\nabla\varphi$$

and

$$\begin{aligned} -H(E_b) &= \nabla(-\nabla E_b)^T \\ &= k\nabla\left((\varphi - \bar{\varphi})\nabla\varphi\right)^T \\ &= k\left((\varphi - \bar{\varphi})H(\varphi) + \nabla\varphi^T\nabla\varphi\right) \end{aligned}$$

where

$$k = -k_b \frac{6\|\bar{\mathbf{e}}_0\|^2}{\bar{A}}, \quad \varphi = \varphi(\theta), \quad \bar{\varphi} = \varphi(\bar{\theta})$$

For $\varphi(\theta) = \theta$ as in Equation (1) we simply have $\nabla\varphi(\theta) = \nabla\theta$ and $H(\varphi(\theta)) = H(\theta)$, and thus

$$-\nabla E_b = k(\theta - \bar{\theta})\nabla\theta$$
$$-H(E_b) = k(\theta - \bar{\theta})H(\theta) + k\nabla\theta^T\nabla\theta$$

As an alternative we might consider the discrete curvature measure given by Equation (6). For convenience and to make it easier to handle the powers of 2, let

$$\psi(\theta) = \tan\left(\frac{\theta}{2^n}\right)$$

such that $\varphi(\theta) = 2^n\psi(\theta)$. Since

$$\nabla\psi = \left(1 + \tan^2\frac{\theta}{2^n}\right)\frac{1}{2^n}\nabla\theta = 2^{-n}(1 + \psi^2)\nabla\theta$$

it follows that

$$\nabla\varphi = \nabla(2^n\psi) = 2^n\nabla\psi = (1 + \psi^2)\nabla\theta$$

and therefore

$$
\begin{aligned}
H(\varphi(\theta)) &= \nabla(\nabla\varphi)^T \\
&= \nabla((1+\psi^2)\nabla\theta)^T \\
&= (1+\psi^2)H(\theta) + \nabla\theta^T\nabla\psi^2 \\
&= (1+\psi^2)H(\theta) + 2\psi\nabla\theta^T\nabla\psi \\
&= (1+\psi^2)H(\theta) + 2^{1-n}\psi(1+\psi^2)\nabla\theta^T\nabla\theta
\end{aligned}
$$

Combining all this gives

$$
\begin{aligned}
-\nabla E_b &= k(\varphi - \bar\varphi)\nabla\varphi \\
&= k(\varphi - \bar\varphi)(1+\psi^2)\nabla\theta \\
-H(E_b) &= k\left((\varphi - \bar\varphi)H(\varphi) + \nabla\varphi^T\nabla\varphi\right) \\
&= k(\varphi - \bar\varphi)\left((1+\psi^2)H(\theta) + 2^{1-n}\psi(1+\psi^2)\nabla\theta^T\nabla\theta\right) \\
&\quad + k(1+\psi^2)^2\nabla\theta^T\nabla\theta \\
&= k(\varphi - \bar\varphi)(1+\psi^2)H(\theta) \\
&\quad + k(1+\psi^2)\left(2(\psi - \bar\psi)\psi + (1+\psi^2)\right)\nabla\theta^T\nabla\theta
\end{aligned}
$$

If we define

$$
\begin{aligned}
\zeta(\theta,\bar\theta) &= k(\varphi - \bar\varphi)(1+\psi^2) = 2^n k(\psi - \bar\psi)(1+\psi^2) \\
\xi(\theta,\bar\theta) &= k(1+\psi^2)\left(2(\psi - \bar\psi)\psi + (1+\psi^2)\right)
\end{aligned}
$$

then we can write the result compactly as

$$
-\nabla E_b = \zeta(\theta,\bar\theta)\nabla\theta \tag{11}
$$

$$
-H(E_b) = \zeta(\theta,\bar\theta)H(\theta) + \xi(\theta,\bar\theta)\nabla\theta^T\nabla\theta \tag{12}
$$

As we can see, the Hessian of the bending energy is a weighted sum of $H(\theta)$ and $\nabla\theta^T\nabla\theta$, and the same weighting function is used in the computation of the bending force and the Hessian of the bending energy. This was also the case with the choice of $\varphi(\theta) = \theta$, and it provides an opportunity for some computational savings later on. The weighting functions, $\zeta(\theta,\bar\theta)$ and $\xi(\theta,\bar\theta)$ are functions of the bend angle and therefore associated with the edge in the middle of the hinge. It should also be noted that the only dependence on $\bar\theta$ is encapsulated in these two functions. Hence in order to evaluate viscous damping forces as in [Kharevych et al. 06, Bergou et al. 10], you only have to replace $\bar\theta$ in each of these functions with the bend angle from the previous timestep, θ_{prev}. The *combined* forces and force gradients can then be

computed by letting :

$$\zeta = \zeta(\theta, \bar{\theta}) + \zeta(\theta, \theta_{prev})$$
$$\xi = \xi(\theta, \bar{\theta}) + \xi(\theta, \theta_{prev})$$

5.   Gradient of the bend angle

In this section we will focus on how to compute $\nabla\theta$. Since we will need it later on, we start by noticing that $\mathbf{e}_0 = \mathbf{x}_1 - \mathbf{x}_0$, so it's easy to see that

$$\nabla\mathbf{e}_0 = [-I, I, 0, 0]$$

where I and 0 each represent 3×3 block matrices. The gradients of the other edge vectors have similar expressions.

For the derivation of $\nabla\theta$ let $\hat{\mathbf{n}}_1$ and $\hat{\mathbf{n}}_2$ be the unit-length normals of the two triangles :

$$\hat{\mathbf{n}}_1 = \widehat{\mathbf{e}_0 \times \mathbf{e}_3}$$
$$\hat{\mathbf{n}}_2 = -\widehat{\mathbf{e}_0 \times \mathbf{e}_4}$$

We then have

$$\hat{\mathbf{n}}_1 \cdot \hat{\mathbf{n}}_2 = \cos\theta$$

Since $\nabla\cos\theta = -\sin(\theta)\nabla\theta$ it follows that (for $\theta \neq 0$) :

$$\nabla\theta = -\frac{\nabla(\hat{\mathbf{n}}_1 \cdot \hat{\mathbf{n}}_2)}{\sin\theta} \tag{13}$$

Using the dot product rule we have :

$$\nabla(\hat{\mathbf{n}}_1 \cdot \hat{\mathbf{n}}_2) = \hat{\mathbf{n}}_1^T \nabla\hat{\mathbf{n}}_2 + \hat{\mathbf{n}}_2^T \nabla\hat{\mathbf{n}}_1$$

The expressions for $\nabla\hat{\mathbf{n}}_1$ and $\nabla\hat{\mathbf{n}}_2$ follow from Equation (10) :

$$\nabla\hat{\mathbf{n}}_1 = \frac{(\mathbf{e}_0 \times \hat{\mathbf{n}}_1)\hat{\mathbf{n}}_1^T \nabla\mathbf{e}_3 - (\mathbf{e}_3 \times \hat{\mathbf{n}}_1)\hat{\mathbf{n}}_1^T \nabla\mathbf{e}_0}{\|\mathbf{n}_1\|} \tag{14}$$

$$\nabla\hat{\mathbf{n}}_2 = \frac{-(\mathbf{e}_0 \times \hat{\mathbf{n}}_2)\hat{\mathbf{n}}_2^T \nabla\mathbf{e}_4 + (\mathbf{e}_4 \times \hat{\mathbf{n}}_2)\hat{\mathbf{n}}_2^T \nabla\mathbf{e}_0}{\|\mathbf{n}_2\|} \tag{15}$$

By using the vector triple product $[\mathbf{u}, \mathbf{v}, \mathbf{w}] = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \mathbf{w} \cdot (\mathbf{u} \times \mathbf{v})$ and

the fact that $[\mathbf{u}, \mathbf{v}, \mathbf{w}] = [\mathbf{w}, \mathbf{u}, \mathbf{v}]$ we can now write :

$$
\begin{aligned}
\hat{\mathbf{n}}_1^T \nabla \hat{\mathbf{n}}_2 &= \frac{-[\hat{\mathbf{n}}_1, \mathbf{e}_0, \hat{\mathbf{n}}_2]\hat{\mathbf{n}}_2^T \nabla \mathbf{e}_4 + [\hat{\mathbf{n}}_1, \mathbf{e}_4, \hat{\mathbf{n}}_2]\hat{\mathbf{n}}_2^T \nabla \mathbf{e}_0}{\|\mathbf{n}_2\|} \\
&= \frac{[\hat{\mathbf{n}}_1, \hat{\mathbf{n}}_2, \mathbf{e}_0]\hat{\mathbf{n}}_2^T \nabla \mathbf{e}_4 - [\hat{\mathbf{n}}_1, \hat{\mathbf{n}}_2, \mathbf{e}_4]\hat{\mathbf{n}}_2^T \nabla \mathbf{e}_0}{\|\mathbf{n}_2\|} \\
\hat{\mathbf{n}}_2^T \nabla \hat{\mathbf{n}}_1 &= \frac{[\hat{\mathbf{n}}_2, \mathbf{e}_0, \hat{\mathbf{n}}_1]\hat{\mathbf{n}}_1^T \nabla \mathbf{e}_3 - [\hat{\mathbf{n}}_2, \mathbf{e}_3, \hat{\mathbf{n}}_1]\hat{\mathbf{n}}_1^T \nabla \mathbf{e}_0}{\|\mathbf{n}_1\|} \\
&= \frac{[\hat{\mathbf{n}}_1, \hat{\mathbf{n}}_2, \mathbf{e}_0]\hat{\mathbf{n}}_1^T \nabla \mathbf{e}_3 - [\hat{\mathbf{n}}_1, \hat{\mathbf{n}}_2, \mathbf{e}_3]\hat{\mathbf{n}}_1^T \nabla \mathbf{e}_0}{\|\mathbf{n}_1\|}
\end{aligned}
$$

Given that

$$\hat{\mathbf{n}}_1 \times \hat{\mathbf{n}}_2 = \hat{\mathbf{e}}_0 \sin \theta$$

it follows that

$$[\hat{\mathbf{n}}_1, \hat{\mathbf{n}}_2, \mathbf{e}_0] = (\hat{\mathbf{n}}_1 \times \hat{\mathbf{n}}_2) \cdot \mathbf{e}_0 = (\hat{\mathbf{e}}_0 \cdot \mathbf{e}_0) \sin \theta = \|\mathbf{e}_0\| \sin \theta$$

For the two remaining volume products we get :

$$
\begin{aligned}
[\hat{\mathbf{n}}_1, \hat{\mathbf{n}}_2, \mathbf{e}_3] &= (\hat{\mathbf{n}}_1 \times \hat{\mathbf{n}}_2) \cdot \mathbf{e}_3 = (\hat{\mathbf{e}}_0 \cdot \mathbf{e}_3) \sin \theta = -\|\mathbf{e}_3\| \cos \alpha_3 \sin \theta, \\
[\hat{\mathbf{n}}_1, \hat{\mathbf{n}}_2, \mathbf{e}_4] &= (\hat{\mathbf{n}}_1 \times \hat{\mathbf{n}}_2) \cdot \mathbf{e}_4 = (\hat{\mathbf{e}}_0 \cdot \mathbf{e}_4) \sin \theta = -\|\mathbf{e}_4\| \cos \alpha_4 \sin \theta
\end{aligned}
$$

The minus signs here are due to the fact that $\mathbf{e}_3$ and $\mathbf{e}_4$ are oriented oppositely of $\mathbf{e}_0$. From the results we see that all the volume products have a factor of $\sin \theta$ which is going to cancel out with the $\sin \theta$ in equation 13 (and lift the singularity at $\theta = 0$). We therefore have

$$
\begin{aligned}
\frac{\hat{\mathbf{n}}_1^T \nabla \hat{\mathbf{n}}_2}{\sin \theta} &= \frac{\|\mathbf{e}_0\|\hat{\mathbf{n}}_2^T \nabla \mathbf{e}_4 + \|\mathbf{e}_4\| \cos \alpha_4 \hat{\mathbf{n}}_2^T \nabla \mathbf{e}_0}{\|\mathbf{n}_2\|} \\
\frac{\hat{\mathbf{n}}_2^T \nabla \hat{\mathbf{n}}_1}{\sin \theta} &= \frac{\|\mathbf{e}_0\|\hat{\mathbf{n}}_1^T \nabla \mathbf{e}_3 + \|\mathbf{e}_3\| \cos \alpha_3 \hat{\mathbf{n}}_1^T \nabla \mathbf{e}_0}{\|\mathbf{n}_1\|}
\end{aligned}
$$

or equivalently

$$\frac{\hat{\mathbf{n}}_1^T \nabla \hat{\mathbf{n}}_2}{\sin \theta} = \frac{\|\mathbf{e}_0\|\hat{\mathbf{n}}_2^T \nabla \mathbf{e}_4 - (\hat{\mathbf{e}}_0 \cdot \mathbf{e}_4)\hat{\mathbf{n}}_2^T \nabla \mathbf{e}_0}{\|\mathbf{n}_2\|} \tag{16}$$

$$\frac{\hat{\mathbf{n}}_2^T \nabla \hat{\mathbf{n}}_1}{\sin \theta} = \frac{\|\mathbf{e}_0\|\hat{\mathbf{n}}_1^T \nabla \mathbf{e}_3 - (\hat{\mathbf{e}}_0 \cdot \mathbf{e}_3)\hat{\mathbf{n}}_1^T \nabla \mathbf{e}_0}{\|\mathbf{n}_1\|} \tag{17}$$

In terms of computation the second of these expressions is probably the most efficient, and we shall use it later to show the connection between the results in [Grinspun et al. 03] and [Bridson et al. 03].

From a geometric point of view, a few more simplifications can be made. The other quantities we need for this are based on different ways of computing the area of the two triangles :

$$
\begin{aligned}
2A_1 &= \|\mathbf{n}_1\| = \frac{\|\mathbf{e}_0\|^2}{\cot\alpha_1 + \cot\alpha_3} = \|\mathbf{e}_0\|\|\mathbf{e}_3\|\sin\alpha_3 \\
2A_2 &= \|\mathbf{n}_2\| = \frac{\|\mathbf{e}_0\|^2}{\cot\alpha_2 + \cot\alpha_4} = \|\mathbf{e}_0\|\|\mathbf{e}_4\|\sin\alpha_4
\end{aligned}
$$

Applying a combination of these identities to equations 16-17 we get :

$$
\begin{aligned}
\frac{\hat{\mathbf{n}}_1^T \nabla \hat{\mathbf{n}}_2}{\sin\theta} &= \frac{1}{\|\mathbf{e}_0\|}(\cot\alpha_2 + \cot\alpha_4)\hat{\mathbf{n}}_2^T \nabla\mathbf{e}_4 + \frac{1}{\|\mathbf{e}_0\|}\cot\alpha_4 \hat{\mathbf{n}}_2^T \nabla\mathbf{e}_0 \\
\frac{\hat{\mathbf{n}}_2^T \nabla \hat{\mathbf{n}}_1}{\sin\theta} &= \frac{1}{\|\mathbf{e}_0\|}(\cot\alpha_1 + \cot\alpha_3)\hat{\mathbf{n}}_1^T \nabla\mathbf{e}_3 + \frac{1}{\|\mathbf{e}_0\|}\cot\alpha_3 \hat{\mathbf{n}}_1^T \nabla\mathbf{e}_0
\end{aligned}
$$

Since

$$
\begin{aligned}
\nabla\mathbf{e}_0 &= (-I, I, 0, 0) \\
\nabla\mathbf{e}_3 &= (0, -I, I, 0) \\
\nabla\mathbf{e}_4 &= (0, -I, 0, I)
\end{aligned}
$$

it then follows easily that :

$$
\begin{aligned}
\nabla_{\mathbf{x}_0}\theta &= \frac{1}{\|\mathbf{e}_0\|}(\cot\alpha_3 \hat{\mathbf{n}}_1^T + \cot\alpha_4 \hat{\mathbf{n}}_2^T) \\
\nabla_{\mathbf{x}_1}\theta &= \frac{1}{\|\mathbf{e}_0\|}(\cot\alpha_1 \hat{\mathbf{n}}_1^T + \cot\alpha_2 \hat{\mathbf{n}}_2^T) \\
\nabla_{\mathbf{x}_2}\theta &= \frac{-1}{\|\mathbf{e}_0\|}(\cot\alpha_1 + \cot\alpha_3)\hat{\mathbf{n}}_1^T = -\frac{\|\mathbf{e}_0\|}{2A_1}\hat{\mathbf{n}}_1^T \\
\nabla_{\mathbf{x}_3}\theta &= \frac{-1}{\|\mathbf{e}_0\|}(\cot\alpha_2 + \cot\alpha_4)\hat{\mathbf{n}}_2^T = -\frac{\|\mathbf{e}_0\|}{2A_2}\hat{\mathbf{n}}_2^T
\end{aligned}
$$

We see that the gradient wrt. $\mathbf{x}_0$ and $\mathbf{x}_1$ are similar just as the gradient wrt. $\mathbf{x}_2$ and $\mathbf{x}_3$ are similar. This is to be expected due to the symmetry of the configuration. Furthermore we notice that the gradients wrt. $\mathbf{x}_2$ and $\mathbf{x}_3$ are easy to interpret geometrically. In the case of $\nabla_{\mathbf{x}_2}\theta$, it is aligned with $\hat{\mathbf{n}}_1$ since any motion of $\mathbf{x}_2$ in the plane orthogonal to $\hat{\mathbf{n}}_1$ will not change θ. To verify the magnitude of these gradients we note that the areas of the triangles can also be written in terms of a baseline and an altitude :

$$
\begin{aligned}
2A_1 &= \|\mathbf{n}_1\| = \|\mathbf{e}_0\|h_{01} = \|\mathbf{e}_1\|h_1 = \|\mathbf{e}_3\|h_3 \\
2A_2 &= \|\mathbf{n}_2\| = \|\mathbf{e}_0\|h_{02} = \|\mathbf{e}_2\|h_2 = \|\mathbf{e}_4\|h_4
\end{aligned}
$$

Hence we see that

$$\nabla_{\mathbf{x}_2}\theta = -\frac{\hat{\mathbf{n}}_1^T}{h_{01}}$$

$$\nabla_{\mathbf{x}_3}\theta = -\frac{\hat{\mathbf{n}}_2^T}{h_{02}}$$

This is in fact what we would expect, since it is easy to see that for a motion, dx, along $\hat{\mathbf{n}}_1$ of $\mathbf{x}_2$ we have

$$\frac{dx}{h_{01}} = \tan(-d\theta) = -d\theta + O(d\theta^3)$$

To first order the scaling factor of $\nabla_{\mathbf{x}_2}\theta$ must therefore be $-1/h_{01}$ which is exactly what we have. A similar argument holds for the gradient wrt. $\mathbf{x}_3$.

At this point it is worthwhile to do a "unit-check". We know that the length of one of the sides is measured in meters (when using SI units), a unit-vector is dimensionless and so is cotan of an angle. It therefore follows that the unit for the gradient is m^{-1}. If we consider a small displacement $\delta\mathbf{x}$ (which is measured in meters), it follows that $\nabla\theta\delta\mathbf{x}$ is dimensionless. This is as expected because θ is dimensionless.

From equation 16 and 17 we can alternatively write these results as follows :

$$
\begin{aligned}
\nabla_{\mathbf{x}_0}\theta &= -\frac{\hat{\mathbf{e}}_0 \cdot \mathbf{e}_3}{\|\mathbf{n}_1\|}\hat{\mathbf{n}}_1^T - \frac{\hat{\mathbf{e}}_0 \cdot \mathbf{e}_4}{\|\mathbf{n}_2\|}\hat{\mathbf{n}}_2^T \\
&= -\frac{(\mathbf{x}_2 - \mathbf{x}_1) \cdot \mathbf{e}_0}{\|\mathbf{e}_0\|}\frac{\mathbf{n}_1^T}{\|\mathbf{n}_1\|^2} - \frac{(\mathbf{x}_3 - \mathbf{x}_1) \cdot \mathbf{e}_0}{\|\mathbf{e}_0\|}\frac{\mathbf{n}_2^T}{\|\mathbf{n}_2\|^2} \\
\nabla_{\mathbf{x}_1}\theta &= \frac{\|\mathbf{e}_0\|\hat{\mathbf{n}}_1^T + (\hat{\mathbf{e}}_0 \cdot \mathbf{e}_3)\hat{\mathbf{n}}_1^T}{\|\mathbf{n}_1\|} + \frac{\|\mathbf{e}_0\|\hat{\mathbf{n}}_2^T + (\hat{\mathbf{e}}_0 \cdot \mathbf{e}_4)\hat{\mathbf{n}}_2^T}{\|\mathbf{n}_2\|} \\
&= (\|\mathbf{e}_0\| + (\hat{\mathbf{e}}_0 \cdot \mathbf{e}_3))\frac{\hat{\mathbf{n}}_1^T}{\|\mathbf{n}_1\|} + (\|\mathbf{e}_0\| + (\hat{\mathbf{e}}_0 \cdot \mathbf{e}_4))\frac{\hat{\mathbf{n}}_2^T}{\|\mathbf{n}_2\|} \\
&= (\hat{\mathbf{e}}_0 \cdot (\mathbf{e}_0 + \mathbf{e}_3))\frac{\hat{\mathbf{n}}_1^T}{\|\mathbf{n}_1\|} + (\hat{\mathbf{e}}_0 \cdot (\mathbf{e}_0 + \mathbf{e}_4))\frac{\hat{\mathbf{n}}_2^T}{\|\mathbf{n}_2\|} \\
&= \frac{(\mathbf{x}_2 - \mathbf{x}_0) \cdot \mathbf{e}_0}{\|\mathbf{e}_0\|}\frac{\mathbf{n}_1^T}{\|\mathbf{n}_1\|^2} + \frac{(\mathbf{x}_3 - \mathbf{x}_0) \cdot \mathbf{e}_0}{\|\mathbf{e}_0\|}\frac{\mathbf{n}_2^T}{\|\mathbf{n}_2\|^2} \\
\nabla_{\mathbf{x}_2}\theta &= -\frac{\|\mathbf{e}_0\|\hat{\mathbf{n}}_1^T}{\|\mathbf{n}_1\|} = -\|\mathbf{e}_0\|\frac{\mathbf{n}_1^T}{\|\mathbf{n}_1\|^2} \\
\nabla_{\mathbf{x}_3}\theta &= -\frac{\|\mathbf{e}_0\|\hat{\mathbf{n}}_2^T}{\|\mathbf{n}_2\|} = -\|\mathbf{e}_0\|\frac{\mathbf{n}_2^T}{\|\mathbf{n}_2\|^2}
\end{aligned}
$$

Except for a scaling factor of -1 each of these expressions correspond exactly to the formulation presented in [Bridson et al. 03]. Using the notation from

[Bridson et al. 03] we have $u_1 = -\nabla_{\mathbf{x}_2}\theta$, $u_2 = -\nabla_{\mathbf{x}_3}\theta$, $u_3 = -\nabla_{\mathbf{x}_0}\theta$, and $u_4 = -\nabla_{\mathbf{x}_1}\theta$. As before we notice the symmetry in the above expressions which requires that the expressions for $\nabla_{\mathbf{x}_0}\theta$ and $\nabla_{\mathbf{x}_1}\theta$ differ by a sign due to the fixed orientation of $\hat{\mathbf{e}}_0$.

For future reference let's summarize the vector formulation of the gradients here :

$$
\begin{aligned}
\nabla_{\mathbf{x}_0}\theta &= -\frac{\hat{\mathbf{e}}_0 \cdot \mathbf{e}_3}{\|\mathbf{n}_1\|}\hat{\mathbf{n}}_1^T - \frac{\hat{\mathbf{e}}_0 \cdot \mathbf{e}_4}{\|\mathbf{n}_2\|}\hat{\mathbf{n}}_2^T \\
\nabla_{\mathbf{x}_1}\theta &= \frac{\hat{\mathbf{e}}_0 \cdot \mathbf{e}_1}{\|\mathbf{n}_1\|}\hat{\mathbf{n}}_1^T + \frac{\hat{\mathbf{e}}_0 \cdot \mathbf{e}_2}{\|\mathbf{n}_2\|}\hat{\mathbf{n}}_2^T \\
\nabla_{\mathbf{x}_2}\theta &= -\frac{\|\mathbf{e}_0\|}{\|\mathbf{n}_1\|}\hat{\mathbf{n}}_1^T \\
\nabla_{\mathbf{x}_3}\theta &= -\frac{\|\mathbf{e}_0\|}{\|\mathbf{n}_2\|}\hat{\mathbf{n}}_2^T
\end{aligned}
$$

The units of these expressions are also as we would expect when we notice that $\|\mathbf{n}\|$ measures an area. It's the magnitude of the product of two vectors, each of which is measured in meters, so its unit must be m^2. Once again the units for $\nabla\theta$ therefore becomes m^{-1}.

The last variation of the above result can be obtained by using that $\hat{\mathbf{e}}_0 \cdot \mathbf{e}_3 = -\|\mathbf{e}_3\| \cos\alpha_3$ and similar for the other dot products. Combined with the area expressions based on height and baseline this gives :

$$\nabla_{\mathbf{x}_0}\theta = \frac{\cos\alpha_3}{h_3}\hat{\mathbf{n}}_1^T + \frac{\cos\alpha_4}{h_4}\hat{\mathbf{n}}_2^T \tag{18}$$

$$\nabla_{\mathbf{x}_1}\theta = \frac{\cos\alpha_1}{h_1}\hat{\mathbf{n}}_1^T + \frac{\cos\alpha_2}{h_2}\hat{\mathbf{n}}_2^T \tag{19}$$

$$\nabla_{\mathbf{x}_2}\theta = -\frac{1}{h_{01}}\hat{\mathbf{n}}_1^T \tag{20}$$

$$\nabla_{\mathbf{x}_3}\theta = -\frac{1}{h_{02}}\hat{\mathbf{n}}_2^T \tag{21}$$

6.   The Hessian of the bend angle

The gradient of the bending forces is given by minus the Hessian of the bending energy. To compute this we clearly need the Hessian of the bend-angle :

$$H(\theta) = \nabla(\nabla\theta)^T$$

By construction this entire matrix is symmetric since the order of differentiation in mixed derivatives doesn't matter. By switching the order of differentiation and enforcing symmetry we also get :

$$\nabla_{\mathbf{x}_i}(\nabla_{\mathbf{x}_j}\theta)^T = (\nabla_{\mathbf{x}_j}(\nabla_{\mathbf{x}_i}\theta)^T)^T$$

In particular that means that the block-diagonals must be symmetric themselves.

To more easily manage the expressions we will compute the elements one block-row at a time, i.e., we will compute $\nabla(\nabla_{\mathbf{x}_i}\theta)^T$ for $i \in \{0, 1, 2, 3\}$. Clearly the results for $i = 1$ and $i = 3$ are related to the corresponding results for $i = 0$ and $i = 2$. We shall use that as a sanity check in the following.

In the computations we will need $\nabla\hat{\mathbf{n}}_1$ and $\nabla\hat{\mathbf{n}}_2$ which were computed in equation 14 and 15. However, they are worth revisiting. Using the inplane edge normals shown in Figure 2 we see that

$$
\begin{aligned}
\nabla\hat{\mathbf{n}}_1 &= \frac{(\mathbf{e}_0 \times \hat{\mathbf{n}}_1)\hat{\mathbf{n}}_1^T\nabla\mathbf{e}_3 - (\mathbf{e}_3 \times \hat{\mathbf{n}}_1)\hat{\mathbf{n}}_1^T\nabla\mathbf{e}_0}{\|\mathbf{n}_1\|} \\
&= \frac{\|\mathbf{e}_0\|(\hat{\mathbf{e}}_0 \times \hat{\mathbf{n}}_1)}{\|\mathbf{n}_1\|}\hat{\mathbf{n}}_1^T\nabla\mathbf{e}_3 - \frac{\|\mathbf{e}_3\|(\hat{\mathbf{e}}_3 \times \hat{\mathbf{n}}_1)}{\|\mathbf{n}_1\|}\hat{\mathbf{n}}_1^T\nabla\mathbf{e}_0 \\
&= \frac{\|\mathbf{e}_0\|\hat{\mathbf{m}}_{01}}{\|\mathbf{n}_1\|}\hat{\mathbf{n}}_1^T\nabla\mathbf{e}_3 - \frac{\|\mathbf{e}_3\|\hat{\mathbf{m}}_3}{\|\mathbf{n}_1\|}\hat{\mathbf{n}}_1^T\nabla\mathbf{e}_0 \\
&= \frac{\hat{\mathbf{m}}_{01}}{h_{01}}\hat{\mathbf{n}}_1^T\nabla\mathbf{e}_3 - \frac{\hat{\mathbf{m}}_3}{h_3}\hat{\mathbf{n}}_1^T\nabla\mathbf{e}_0
\end{aligned}
$$

When we break this into blocks we get

$$
\nabla_{\mathbf{x}_0}\hat{\mathbf{n}}_1 = \frac{\hat{\mathbf{m}}_3}{h_3}\hat{\mathbf{n}}_1^T \tag{22}
$$

$$
\nabla_{\mathbf{x}_1}\hat{\mathbf{n}}_1 = -\frac{\|\mathbf{e}_0\|\hat{\mathbf{m}}_{01} + \|\mathbf{e}_3\|\hat{\mathbf{m}}_3}{\|\mathbf{n}_1\|}\hat{\mathbf{n}}_1^T = -\frac{\mathbf{m}_{01} + \mathbf{m}_3}{\|\mathbf{n}_1\|}\hat{\mathbf{n}}_1^T
$$

$$
= \frac{\mathbf{m}_1}{\|\mathbf{n}_1\|}\hat{\mathbf{n}}_1^T = \frac{\|\mathbf{e}_1\|\hat{\mathbf{m}}_1}{\|\mathbf{n}_1\|}\hat{\mathbf{n}}_1^T = \frac{\hat{\mathbf{m}}_1}{h_1}\hat{\mathbf{n}}_1^T \tag{23}
$$

$$
\nabla_{\mathbf{x}_2}\hat{\mathbf{n}}_1 = \frac{\hat{\mathbf{m}}_{01}}{h_{01}}\hat{\mathbf{n}}_1^T \tag{24}
$$

$$
\nabla_{\mathbf{x}_3}\hat{\mathbf{n}}_1 = 0 \tag{25}
$$

The pattern should now be fairly obvious. The gradient of the normal with respect to a vertex is the outer product of the opposing edge normal with the normal itself and scaled with one over the height to the opposing edge. Based on this we can immediately write the gradient of $\hat{\mathbf{n}}_2$:

$$
\begin{aligned}
\nabla_{\mathbf{x}_0}\hat{\mathbf{n}}_2 &= \frac{\hat{\mathbf{m}}_4}{h_4}\hat{\mathbf{n}}_2^T \\
\nabla_{\mathbf{x}_1}\hat{\mathbf{n}}_2 &= \frac{\hat{\mathbf{m}}_2}{h_2}\hat{\mathbf{n}}_2^T \\
\nabla_{\mathbf{x}_2}\hat{\mathbf{n}}_2 &= 0 \\
\nabla_{\mathbf{x}_3}\hat{\mathbf{n}}_2 &= \frac{\hat{\mathbf{m}}_{02}}{h_{02}}\hat{\mathbf{n}}_2^T
\end{aligned}
$$

Intuitively this is not surprising. Given a small displacement $\delta \mathbf{x}$ we see that any component orthogonal to the normal will be projected away, and the remaining component will cause the normal to tilt toward or away from the opposing edge (depending on whether the vertex is being moved up or down). The scaling factor is the same as we saw in the previous section when computing $\nabla \theta$ and for the same reason.

With these building blocks in place, we now notice that for all the gradients of θ we have a scalar function times a vector, so the gradient of that is given by equation 10. Starting with the simpler expression we get the following for $\nabla(\nabla_{\mathbf{x}_2}\theta)^T$:

$$
\begin{aligned}
\nabla(\nabla_{\mathbf{x}_2}\theta)^T &= \nabla\left(-\frac{\|\mathbf{e}_0\|}{\|\mathbf{n}_1\|}\hat{\mathbf{n}}_1\right) \\
&= -\frac{\|\mathbf{e}_0\|}{\|\mathbf{n}_1\|}\nabla\hat{\mathbf{n}}_1 - \hat{\mathbf{n}}_1\nabla\left(\frac{\|\mathbf{e}_0\|}{\|\mathbf{n}_1\|}\right)
\end{aligned}
\tag{26}
$$

so we just need :

$$
\nabla\left(\frac{\|\mathbf{e}_0\|}{\|\mathbf{n}_1\|}\right) = \nabla\left(\frac{1}{h_{01}}\right) = -\frac{\nabla h_{01}}{h_{01}^2}
\tag{27}
$$

From a geometrical point of view it is easy to see that $\nabla_{\mathbf{x}_2}h_{01}$ must be aligned with $\mathbf{m}_{01}^T$, and in fact we must have $\nabla_{\mathbf{x}_2}h_{01} = -\hat{\mathbf{m}}_{01}^T$ such that

$$
\nabla_{\mathbf{x}_2}\left(\frac{\|\mathbf{e}_0\|}{\|\mathbf{n}_1\|}\right) = \frac{\hat{\mathbf{m}}_{01}^T}{h_{01}^2}
\tag{28}
$$

Similarly we must have $\nabla_{\mathbf{x}_3}h_{01} = 0$. From an algebraic point of view we have :

$$
\begin{aligned}
\nabla\left(\frac{\|\mathbf{e}_0\|}{\|\mathbf{n}_1\|}\right) &= \|\mathbf{e}_0\|\nabla\left(\frac{1}{\|\mathbf{n}_1\|}\right) + \frac{1}{\|\mathbf{n}_1\|}\nabla\|\mathbf{e}_0\| \\
&= -\|\mathbf{e}_0\|\frac{\hat{\mathbf{n}}_1^T\nabla\mathbf{n}_1}{\|\mathbf{n}_1\|^2} + \frac{1}{\|\mathbf{n}_1\|}\hat{\mathbf{e}}_0^T\nabla\mathbf{e}_0 \\
&= \frac{\|\mathbf{n}_1\|\hat{\mathbf{e}}_0^T\nabla\mathbf{e}_0 - \|\mathbf{e}_0\|\hat{\mathbf{n}}_1^T\nabla\mathbf{n}_1}{\|\mathbf{n}_1\|^2} \\
&= \frac{\|\mathbf{n}_1\|\hat{\mathbf{e}}_0^T\nabla\mathbf{e}_0 - \|\mathbf{e}_0\|\hat{\mathbf{n}}_1^T\nabla(\mathbf{e}_0\times\mathbf{e}_3)}{\|\mathbf{n}_1\|^2} \\
&= \frac{\|\mathbf{n}_1\|\hat{\mathbf{e}}_0^T\nabla\mathbf{e}_0 - \|\mathbf{e}_0\|\hat{\mathbf{n}}_1^T(\mathbf{e}_0\times\nabla\mathbf{e}_3 - \mathbf{e}_3\times\nabla\mathbf{e}_0)}{\|\mathbf{n}_1\|^2}
\end{aligned}
$$

In particular we see from this that

$$
\nabla_{\mathbf{x}_2}\left(\frac{\|\mathbf{e}_0\|}{\|\mathbf{n}_1\|}\right) = -\frac{\|\mathbf{e}_0\|\hat{\mathbf{n}}_1^T[\mathbf{e}_0]_\times}{\|\mathbf{n}_1\|^2} = -\frac{\|\mathbf{e}_0\|^2}{\|\mathbf{n}_1\|^2}\hat{\mathbf{n}}_1^T[\hat{\mathbf{e}}_0]_\times = -\frac{\hat{\mathbf{n}}_1^T[\hat{\mathbf{e}}_0]_\times}{h_{01}^2}
$$

Since the matrix representation for the cross product is skew symmetric it follows that

$$\left(\mathbf{u}^T[\mathbf{v}]_\times\right)^T = [\mathbf{v}]_\times^T \mathbf{u} = -[\mathbf{v}]_\times \mathbf{u} = -\mathbf{v} \times \mathbf{u}$$

or

$$\mathbf{u}^T[\mathbf{v}]_\times = -(\mathbf{v} \times \mathbf{u})^T$$

As expected we therefore get

$$\nabla_{\mathbf{x}_2}\left(\frac{\|\mathbf{e}_0\|}{\|\mathbf{n}_1\|}\right) = \frac{(\hat{\mathbf{e}}_0 \times \hat{\mathbf{n}}_1)^T}{h_{01}^2} = \frac{\hat{\mathbf{m}}_{01}^T}{h_{01}^2}$$

The gradient wrt. $\mathbf{x}_0$ and $\mathbf{x}_1$ is not as easy to derive geometrically. However, for both of these it is easy to see that any motion along $\mathbf{e}_0$ will not change h_{01}. Consequently the gradient must be perpendicular to $\mathbf{e}_0$, and in order to increase h_{01} we must move away from $\mathbf{x}_2$, so the gradient must be aligned with $\mathbf{m}_{01}$.

Using the algebraic expression we get

$$\begin{aligned}
\nabla_{\mathbf{x}_0}\left(\frac{\|\mathbf{e}_0\|}{\|\mathbf{n}_1\|}\right) &= -\frac{\hat{\mathbf{e}}_0^T}{\|\mathbf{n}_1\|} - \frac{\|\mathbf{e}_0\|\hat{\mathbf{n}}_1^T[\mathbf{e}_3]_\times}{\|\mathbf{n}_1\|^2} \\
&= -\frac{\hat{\mathbf{e}}_0^T}{\|\mathbf{n}_1\|} + \frac{\|\mathbf{e}_0\|}{\|\mathbf{n}_1\|^2}(\mathbf{e}_3 \times \hat{\mathbf{n}}_1)^T \\
&= -\frac{\hat{\mathbf{e}}_0^T}{\|\mathbf{n}_1\|} + \frac{\|\mathbf{e}_0\|\|\mathbf{e}_3\|}{\|\mathbf{n}_1\|^2}(\hat{\mathbf{e}}_3 \times \hat{\mathbf{n}}_1)^T \\
&= -\frac{\hat{\mathbf{e}}_0^T}{\|\mathbf{n}_1\|} + \frac{\|\mathbf{e}_0\|\|\mathbf{e}_3\|}{\|\mathbf{n}_1\|^2}\hat{\mathbf{m}}_3^T \\
&= -\frac{\hat{\mathbf{e}}_0^T}{\|\mathbf{n}_1\|} + \frac{\hat{\mathbf{m}}_3^T}{h_{01}h_3}
\end{aligned} \tag{29}$$

Since we expect the result to be aligned with $\mathbf{m}_{01}$ we decompose $\mathbf{m}_3$ into a component parallel to $\mathbf{e}_0$ and an orthogonal component parallel to $\mathbf{m}_{01}$. From Figure 4 we get

$$\begin{aligned}
\hat{\mathbf{m}}_3 &= \hat{\mathbf{e}}_0 \cos\left(\frac{\pi}{2} - \alpha_3\right) - \hat{\mathbf{m}}_{01} \sin\left(\frac{\pi}{2} - \alpha_3\right) \\
&= \hat{\mathbf{e}}_0 \sin\alpha_3 - \hat{\mathbf{m}}_{01} \cos\alpha_3
\end{aligned}$$

Inserting this in Equation 29 gives :

$$\begin{aligned}
\nabla_{\mathbf{x}_0}\left(\frac{\|\mathbf{e}_0\|}{\|\mathbf{n}_1\|}\right) &= -\frac{\hat{\mathbf{e}}_0^T}{\|\mathbf{n}_1\|} + \frac{\hat{\mathbf{m}}_3^T}{h_{01}h_3} \\
&= \left(-\frac{1}{\|\mathbf{n}_1\|} + \frac{\sin\alpha_3}{h_{01}h_3}\right)\hat{\mathbf{e}}_0^T - \frac{\cos\alpha_3}{h_{01}h_3}\hat{\mathbf{m}}_{01}^T
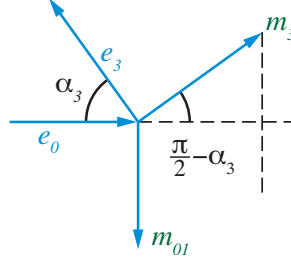\end{aligned}$$

Figure 4. *The decomposition of* $\mathbf{m}_3$ *along* $\mathbf{e}_0$ *and* $\mathbf{m}_{01}$.

However, since $\sin\alpha_3 = h_3/\|\mathbf{e}_0\|$ we see that

$$-\frac{1}{\|\mathbf{n}_1\|} + \frac{\sin\alpha_3}{h_{01}h_3} = -\frac{1}{h_{01}\|\mathbf{e}_0\|} + \frac{h_3/\|\mathbf{e}_0\|}{h_{01}h_3} = 0$$

and therefore

$$
\begin{aligned}
\nabla_{\mathbf{x}_0}\left(\frac{\|\mathbf{e}_0\|}{\|\mathbf{n}_1\|}\right) &= -\frac{\cos\alpha_3}{h_{01}h_3}\hat{\mathbf{m}}_{01}^T \\
&= \frac{\hat{\mathbf{e}}_0 \cdot \hat{\mathbf{e}}_3}{h_{01}h_3}\hat{\mathbf{m}}_{01}^T \\
&= \frac{\|\mathbf{e}_0\|\|\mathbf{e}_3\|\hat{\mathbf{e}}_0 \cdot \hat{\mathbf{e}}_3}{\|\mathbf{n}_1\|^2}\hat{\mathbf{m}}_{01}^T \\
&= \frac{\mathbf{e}_0 \cdot \mathbf{e}_3}{\|\mathbf{n}_1\|^2}\hat{\mathbf{m}}_{01}^T
\end{aligned}
$$

By symmetry we must have

$$\nabla_{\mathbf{x}_1}\left(\frac{\|\mathbf{e}_0\|}{\|\mathbf{n}_1\|}\right) = -\frac{\cos\alpha_1}{h_{01}h_1}\hat{\mathbf{m}}_{01}^T = -\frac{\mathbf{e}_0 \cdot \mathbf{e}_1}{\|\mathbf{n}_1\|^2}\hat{\mathbf{m}}_{01}^T$$

In summary this gives us

$$\nabla_{\mathbf{x}_0}\left(\frac{\|\mathbf{e}_0\|}{\|\mathbf{n}_1\|}\right) = \frac{\mathbf{e}_0 \cdot \mathbf{e}_3}{\|\mathbf{n}_1\|^2}\hat{\mathbf{m}}_{01}^T \quad = -\frac{\cos\alpha_3}{h_{01}h_3}\hat{\mathbf{m}}_{01}^T \tag{30}$$

$$\nabla_{\mathbf{x}_1}\left(\frac{\|\mathbf{e}_0\|}{\|\mathbf{n}_1\|}\right) = -\frac{\mathbf{e}_0 \cdot \mathbf{e}_1}{\|\mathbf{n}_1\|^2}\hat{\mathbf{m}}_{01}^T = -\frac{\cos\alpha_1}{h_{01}h_1}\hat{\mathbf{m}}_{01}^T \tag{31}$$

$$\nabla_{\mathbf{x}_2}\left(\frac{\|\mathbf{e}_0\|}{\|\mathbf{n}_1\|}\right) = \frac{\hat{\mathbf{m}}_{01}^T}{h_{01}^2} \qquad = \frac{1}{h_{01}^2}\hat{\mathbf{m}}_{01}^T \tag{32}$$

$$\nabla_{\mathbf{x}_3}\left(\frac{\|\mathbf{e}_0\|}{\|\mathbf{n}_1\|}\right) = 0 \tag{33}$$

By combining these results with Equation 27 we can also read off the expressions for ∇h_{01} :

$$\nabla_{\mathbf{x}_0} h_{01} = \frac{h_{01}}{h_3} \cos \alpha_3 \hat{\mathbf{m}}_{01}^T$$

$$\nabla_{\mathbf{x}_1} h_{01} = \frac{h_{01}}{h_1} \cos \alpha_1 \hat{\mathbf{m}}_{01}^T$$

$$\nabla_{\mathbf{x}_2} h_{01} = -\hat{\mathbf{m}}_{01}^T$$

$$\nabla_{\mathbf{x}_3} h_{01} = 0$$

What's important about this is that it allows us to easily write down the gradients for the heights extending from the other sides of the triangle. Simply rotate all the labels on the triangle and substitute.

Combining equations 30-33 and equations 22-25 with equation 26 we get :

$$\nabla_{\mathbf{x}_0} (\nabla_{\mathbf{x}_2} \theta)^T = -\frac{\|\mathbf{e}_0\| \hat{\mathbf{m}}_3}{\|\mathbf{n}_1\| h_3} \hat{\mathbf{n}}_1^T + \hat{\mathbf{n}}_1 \left(\frac{\cos \alpha_3}{h_{01} h_3} \hat{\mathbf{m}}_{01}^T \right)$$

$$= -\frac{1}{h_{01} h_3} \left(\hat{\mathbf{m}}_3 \hat{\mathbf{n}}_1^T - \cos \alpha_3 \hat{\mathbf{n}}_1 \hat{\mathbf{m}}_{01}^T \right) \tag{34}$$

$$\nabla_{\mathbf{x}_1} (\nabla_{\mathbf{x}_2} \theta)^T = -\frac{\|\mathbf{e}_0\| \hat{\mathbf{m}}_1}{\|\mathbf{n}_1\| h_1} \hat{\mathbf{n}}_1^T + \hat{\mathbf{n}}_1 \left(\frac{\cos \alpha_1}{h_{01} h_1} \hat{\mathbf{m}}_{01}^T \right)$$

$$= -\frac{1}{h_{01} h_1} \left(\hat{\mathbf{m}}_1 \hat{\mathbf{n}}_1^T - \cos \alpha_1 \hat{\mathbf{n}}_1 \hat{\mathbf{m}}_{01}^T \right) \tag{35}$$

$$\nabla_{\mathbf{x}_2} (\nabla_{\mathbf{x}_2} \theta)^T = -\frac{\|\mathbf{e}_0\| \mathbf{m}_{01}}{\|\mathbf{n}_1\| h_{01}} \hat{\mathbf{n}}_1^T - \hat{\mathbf{n}}_1 \frac{\hat{\mathbf{m}}_{01}^T}{h_{01}^2}$$

$$= -\frac{1}{h_{01}^2} \left(\hat{\mathbf{m}}_{01} \hat{\mathbf{n}}_1^T + \hat{\mathbf{n}}_1 \hat{\mathbf{m}}_{01}^T \right) \tag{36}$$

$$\nabla_{\mathbf{x}_3} (\nabla_{\mathbf{x}_2} \theta)^T = 0 \tag{37}$$

The results for the other triangle follow by simple substitution :

$$\nabla_{\mathbf{x}_0} (\nabla_{\mathbf{x}_3} \theta)^T = -\frac{1}{h_{02} h_4} \left(\hat{\mathbf{m}}_4 \hat{\mathbf{n}}_2^T - \cos \alpha_4 \hat{\mathbf{n}}_2 \hat{\mathbf{m}}_{02}^T \right) \tag{38}$$

$$\nabla_{\mathbf{x}_1} (\nabla_{\mathbf{x}_3} \theta)^T = -\frac{1}{h_{02} h_2} \left(\hat{\mathbf{m}}_2 \hat{\mathbf{n}}_2^T - \cos \alpha_2 \hat{\mathbf{n}}_2 \hat{\mathbf{m}}_{02}^T \right) \tag{39}$$

$$\nabla_{\mathbf{x}_2} (\nabla_{\mathbf{x}_3} \theta)^T = 0 \tag{40}$$

$$\nabla_{\mathbf{x}_3} (\nabla_{\mathbf{x}_3} \theta)^T = -\frac{1}{h_{02}^2} \left(\hat{\mathbf{m}}_{02} \hat{\mathbf{n}}_2^T + \hat{\mathbf{n}}_2 \hat{\mathbf{m}}_{02}^T \right) \tag{41}$$

At this point it should be noted that the two diagonal blocks $\nabla_{\mathbf{x}_2} (\nabla_{\mathbf{x}_2} \theta)^T$ and $\nabla_{\mathbf{x}_3} (\nabla_{\mathbf{x}_3} \theta)^T$ are in fact symmetric as we would expect.

Moving on to the first row of the Hessian we have from equation 18 :

$$\nabla(\nabla_{\mathbf{x}_0}\theta)^T = \nabla\left(\frac{\cos\alpha_3}{h_3}\hat{\mathbf{n}}_1 + \frac{\cos\alpha_4}{h_4}\hat{\mathbf{n}}_2\right) \tag{42}$$

To compute this we first need

$$\nabla\left(\frac{\cos\alpha_3}{h_3}\hat{\mathbf{n}}_1\right) = \frac{\cos\alpha_3}{h_3}\nabla\hat{\mathbf{n}}_1 + \hat{\mathbf{n}}_1\nabla\left(\frac{\cos\alpha_3}{h_3}\right) \tag{43}$$

and to compute that we need

$$\nabla\left(\frac{\cos\alpha_3}{h_3}\right) = \cos\alpha_3\nabla\left(\frac{1}{h_3}\right) + \frac{1}{h_3}\nabla\cos\alpha_3$$

$$= -\frac{\cos\alpha_3}{h_3^2}\nabla h_3 + \frac{1}{h_3}\nabla\cos\alpha_3 \tag{44}$$

Based on our previous results, the expressions for ∇h_3 are straight forward :

$$\nabla_{\mathbf{x}_0}h_3 = -\hat{\mathbf{m}}_3^T \tag{45}$$

$$\nabla_{\mathbf{x}_1}h_3 = \frac{h_3}{h_1}\cos\beta\hat{\mathbf{m}}_3^T = \frac{h_3}{h_1}(\hat{\mathbf{e}}_1\cdot\hat{\mathbf{e}}_3)\mathbf{m}_3^T \tag{46}$$

$$\nabla_{\mathbf{x}_2}h_3 = \frac{h_3}{h_{01}}\cos\alpha_3\hat{\mathbf{m}}_3^T = -\frac{h_3}{h_{01}}(\hat{\mathbf{e}}_0\cdot\hat{\mathbf{e}}_3)\hat{\mathbf{m}}_3^T \tag{47}$$

$$\nabla_{\mathbf{x}_3}h_3 = 0 \tag{48}$$

Here $\beta = \pi - \alpha_1 - \alpha_3$ is the top angle of triangle 1. We also see immediately that the gradient of α_3 with respect to $\mathbf{x}_0$ and $\mathbf{x}_2$ must be aligned with $\mathbf{m}_{01}$ and $\mathbf{m}_3$ respectively, and that the gradient with respect to $\mathbf{x}_3$ must be zero. The gradient of α_3 with respect to $\mathbf{x}_1$ is more complicated. To derive the exact expressions, let us consider $\nabla\cos\alpha_3$:

$$\nabla\cos\alpha_3 = -\nabla(\hat{\mathbf{e}}_0\cdot\hat{\mathbf{e}}_3)$$

$$= -\hat{\mathbf{e}}_0^T\nabla\hat{\mathbf{e}}_3 - \hat{\mathbf{e}}_3\nabla\hat{\mathbf{e}}_0$$

$$= -\hat{\mathbf{e}}_0^T(1 - \hat{\mathbf{e}}_3\hat{\mathbf{e}}_3^T)\frac{\nabla\mathbf{e}_3}{\|\mathbf{e}_3\|} - \hat{\mathbf{e}}_3^T(1 - \hat{\mathbf{e}}_0\hat{\mathbf{e}}_0^T)\frac{\nabla\mathbf{e}_0}{\|\mathbf{e}_0\|}$$

This breaks into blocks in a straightforward way :

$$\nabla_{\mathbf{x}_0}\cos\alpha_3 = \frac{\hat{\mathbf{e}}_3^T(1 - \hat{\mathbf{e}}_0\hat{\mathbf{e}}_0^T)}{\|\mathbf{e}_0\|}$$

$$\nabla_{\mathbf{x}_1}\cos\alpha_3 = -\frac{\hat{\mathbf{e}}_3^T(1 - \hat{\mathbf{e}}_0\hat{\mathbf{e}}_0^T)}{\|\mathbf{e}_0\|} + \frac{\hat{\mathbf{e}}_0^T(1 - \hat{\mathbf{e}}_3\hat{\mathbf{e}}_3^T)}{\|\mathbf{e}_3\|}$$

$$\nabla_{\mathbf{x}_2}\cos\alpha_3 = -\frac{\hat{\mathbf{e}}_0^T(1 - \hat{\mathbf{e}}_3\hat{\mathbf{e}}_3^T)}{\|\mathbf{e}_3\|}$$

$$\nabla_{\mathbf{x}_3}\cos\alpha_3 = 0$$

To simplify these expressions, consider a variation $\delta\mathbf{x}$ and let it be decomposed according to the orthogonal frame consisting of $(\hat{\mathbf{e}}_j, \hat{\mathbf{n}}_1, \hat{\mathbf{m}}_j)$:

$$\delta\mathbf{x} = \delta\mathbf{x}_{\hat{\mathbf{e}}_j} + \delta\mathbf{x}_{\hat{\mathbf{n}}_1} + \delta\mathbf{x}_{\hat{\mathbf{m}}_j} \tag{49}$$

When this variation is applied to the expression $\hat{\mathbf{e}}_i^T(1 - \hat{\mathbf{e}}_j\hat{\mathbf{e}}_j^T)$ we get :

$$\begin{aligned}
\hat{\mathbf{e}}_i^T(1 - \hat{\mathbf{e}}_j\hat{\mathbf{e}}_j^T)\delta\mathbf{x} &= \hat{\mathbf{e}}_i^T(1 - \hat{\mathbf{e}}_j\hat{\mathbf{e}}_j^T)(\delta\mathbf{x}_{\hat{\mathbf{e}}_j} + \delta\mathbf{x}_{\hat{\mathbf{n}}_1} + \delta\mathbf{x}_{\hat{\mathbf{m}}_j}) \\
&= \hat{\mathbf{e}}_i^T(\delta\mathbf{x}_{\hat{\mathbf{n}}_1} + \delta\mathbf{x}_{\hat{\mathbf{m}}_j}) \\
&= \hat{\mathbf{e}}_i^T\delta\mathbf{x}_{\hat{\mathbf{m}}_j}
\end{aligned}$$

The first equality follows because $(1 - \hat{\mathbf{e}}_j\hat{\mathbf{e}}_j^T)$ projects away any component along $\hat{\mathbf{e}}_j$, and the second equality follows because $\hat{\mathbf{e}}_i$ and $\hat{\mathbf{n}}_1$ are orthogonal. Since

$$\begin{aligned}
\hat{\mathbf{e}}_0 &= \sin\alpha_3\hat{\mathbf{m}}_3 - \cos\alpha_3\hat{\mathbf{e}}_3 \\
\hat{\mathbf{e}}_3 &= -\sin\alpha_3\hat{\mathbf{m}}_{01} - \cos\alpha_3\hat{\mathbf{e}}_0
\end{aligned}$$

we can further rewrite the two relevant variations to get

$$\begin{aligned}
\hat{\mathbf{e}}_3^T\delta\mathbf{x}_{\mathbf{m}_{01}} &= -(\sin\alpha_3\hat{\mathbf{m}}_{01}^T + \cos\alpha_3\hat{\mathbf{e}}_0^T)\delta\mathbf{x}_{\mathbf{m}_{01}} \\
&= -\sin\alpha_3\hat{\mathbf{m}}_{01}^T\delta\mathbf{x}_{\mathbf{m}_{01}} \\
&= -\sin\alpha_3\hat{\mathbf{m}}_{01}^T\delta\mathbf{x} \\
&= -\frac{h_{01}}{\|\mathbf{e}_3\|}\hat{\mathbf{m}}_{01}^T\delta\mathbf{x}
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{e}_0^T\delta\mathbf{x}_{\mathbf{m}_3} &= (\sin\alpha_3\hat{\mathbf{m}}_3^T - \cos\alpha_3\hat{\mathbf{e}}_3^T)\delta\mathbf{x}_{\mathbf{m}_3} \\
&= \sin\alpha_3\hat{\mathbf{m}}_3^T\delta\mathbf{x}_{\mathbf{m}_3} \\
&= \sin\alpha_3\hat{\mathbf{m}}_3^T\delta\mathbf{x} \\
&= \frac{h_3}{\|\mathbf{e}_0\|}\hat{\mathbf{m}}_3^T\delta\mathbf{x}
\end{aligned}$$

It therefore follows that

$$\begin{aligned}
\nabla_{\mathbf{x}_0}\cos\alpha_3 &= -\frac{h_{01}}{\|\mathbf{e}_0\|\|\mathbf{e}_3\|}\hat{\mathbf{m}}_{01}^T \\
\nabla_{\mathbf{x}_1}\cos\alpha_3 &= \frac{h_{01}}{\|\mathbf{e}_0\|\|\mathbf{e}_3\|}\hat{\mathbf{m}}_{01}^T + \frac{h_3}{\|\mathbf{e}_0\|\|\mathbf{e}_3\|}\hat{\mathbf{m}}_3^T \\
\nabla_{\mathbf{x}_2}\cos\alpha_3 &= -\frac{h_3}{\|\mathbf{e}_0\|\|\mathbf{e}_3\|}\hat{\mathbf{m}}_3^T \\
\nabla_{\mathbf{x}_3}\cos\alpha_3 &= 0
\end{aligned}$$

It should be noted here that $\cos\alpha_3$ increases when α_3 decreases which explains the signs in the expressions above. When we combine the above with the expressions for ∇h_3 in equations 45-48 we get the following for the gradient wrt. $\mathbf{x}_0$:

$$\nabla_{\mathbf{x}_0}\left(\frac{\cos\alpha_3}{h_3}\right) = \frac{\cos\alpha_3}{h_3^2}\hat{\mathbf{m}}_3^T - \frac{1}{h_3}\frac{h_{01}}{\|\mathbf{e}_0\|\|\mathbf{e}_3\|}\hat{\mathbf{m}}_{01}^T$$

$$= \frac{\cos\alpha_3}{h_3^2}\hat{\mathbf{m}}_3^T - \frac{\hat{\mathbf{m}}_{01}^T}{\|\mathbf{e}_0\|^2}$$

For the gradient wrt. $\mathbf{x}_1$ we have :

$$\nabla_{\mathbf{x}_1}\left(\frac{\cos\alpha_3}{h_3}\right) = -\frac{\cos\alpha_3}{h_3^2}\frac{h_3}{h_1}(\hat{\mathbf{e}}_1\cdot\hat{\mathbf{e}}_3)\hat{\mathbf{m}}_3^T + \frac{1}{h_3}\frac{1}{\|\mathbf{e}_0\|\|\mathbf{e}_3\|}\left(h_{01}\hat{\mathbf{m}}_{01}^T + h_3\hat{\mathbf{m}}_3^T\right)$$

$$= -\left(\frac{\cos\alpha_3}{h_1 h_3}(\hat{\mathbf{e}}_1\cdot\hat{\mathbf{e}}_3) - \frac{1}{\|\mathbf{e}_0\|\|\mathbf{e}_3\|}\right)\hat{\mathbf{m}}_3^T + \frac{h_{01}}{h_3\|\mathbf{e}_0\|\|\mathbf{e}_3\|}\hat{\mathbf{m}}_{01}^T$$

$$= -\left(\frac{\cos\alpha_3}{h_1 h_3}(\hat{\mathbf{e}}_1\cdot\hat{\mathbf{e}}_3) - \frac{1}{\|\mathbf{e}_0\|\|\mathbf{e}_3\|}\right)\hat{\mathbf{m}}_3^T + \frac{\hat{\mathbf{m}}_{01}^T}{\|\mathbf{e}_0\|^2}$$

$$= -\frac{1}{h_1 h_3}\left(\cos\alpha_3\cos\beta - \frac{h_1 h_3}{\|\mathbf{e}_0\|\|\mathbf{e}_3\|}\right)\hat{\mathbf{m}}_3^T + \frac{\hat{\mathbf{m}}_{01}^T}{\|\mathbf{e}_0\|^2}$$

$$= -\frac{1}{h_1 h_3}\left(\cos\alpha_3\cos\beta - \sin\alpha_3\sin\beta\right)\hat{\mathbf{m}}_3^T + \frac{\hat{\mathbf{m}}_{01}^T}{\|\mathbf{e}_0\|^2}$$

$$= -\frac{1}{h_1 h_3}\cos(\alpha_3 + \beta)\hat{\mathbf{m}}_3^T + \frac{\hat{\mathbf{m}}_{01}^T}{\|\mathbf{e}_0\|^2}$$

$$= -\frac{1}{h_1 h_3}\cos(\pi - \alpha_1)\hat{\mathbf{m}}_3^T + \frac{\hat{\mathbf{m}}_{01}^T}{\|\mathbf{e}_0\|^2}$$

$$= \frac{\cos\alpha_1}{h_1 h_3}\hat{\mathbf{m}}_3^T + \frac{\hat{\mathbf{m}}_{01}^T}{\|\mathbf{e}_0\|^2}$$

Finally, the gradient wrt. $\mathbf{x}_2$ becomes :

$$\nabla_{\mathbf{x}_2}\left(\frac{\cos\alpha_3}{h_3}\right) = \frac{\cos\alpha_3}{h_3^2}\frac{h_3}{h_{01}}(\hat{\mathbf{e}}_0\cdot\hat{\mathbf{e}}_3)\hat{\mathbf{m}}_3^T - \frac{1}{h_3}\frac{h_3}{\|\mathbf{e}_0\|\|\mathbf{e}_3\|}\hat{\mathbf{m}}_3^T$$

$$= -\left(\frac{\cos^2\alpha_3}{h_3 h_{01}} + \frac{1}{\|\mathbf{e}_0\|\|\mathbf{e}_3\|}\right)\hat{\mathbf{m}}_3^T$$

$$= -\frac{1}{h_{01}h_3}\left(\cos^2\alpha_3 + \frac{h_{01}h_3}{\|\mathbf{e}_0\|\|\mathbf{e}_3\|}\right)\hat{\mathbf{m}}_3^T$$

$$= -\frac{1}{h_{01}h_3}\left(\cos^2\alpha_3 + \sin^2\alpha_3\right)\hat{\mathbf{m}}_3^T$$

$$= -\frac{\hat{\mathbf{m}}_3^T}{h_{01}h_3}$$

As previously mentioned the gradient wrt. $\mathbf{x}_3$ is simply 0. In summary we therefore have :

$$\nabla_{\mathbf{x}_0}\left(\frac{\cos\alpha_3}{h_3}\right) = \frac{\cos\alpha_3}{h_3^2}\hat{\mathbf{m}}_3^T - \frac{\hat{\mathbf{m}}_{01}^T}{\|\mathbf{e}_0\|^2}$$

$$\nabla_{\mathbf{x}_1}\left(\frac{\cos\alpha_3}{h_3}\right) = \frac{\cos\alpha_1}{h_1 h_3}\hat{\mathbf{m}}_3^T + \frac{\hat{\mathbf{m}}_{01}^T}{\|\mathbf{e}_0\|^2}$$

$$\nabla_{\mathbf{x}_2}\left(\frac{\cos\alpha_3}{h_3}\right) = -\frac{\hat{\mathbf{m}}_3^T}{h_{01} h_3}$$

$$\nabla_{\mathbf{x}_3}\left(\frac{\cos\alpha_3}{h_3}\right) = 0$$

$$(50)$$

When we insert these results in Equation 43 along with the results from Equation 22-25 we now get :

$$\begin{aligned}
\nabla_{\mathbf{x}_0}\left(\frac{\cos\alpha_3}{h_3}\hat{\mathbf{n}}_1\right) &= \frac{\cos\alpha_3}{h_3}\frac{\hat{\mathbf{m}}_3}{h_3}\hat{\mathbf{n}}_1^T + \hat{\mathbf{n}}_1\left(\frac{\cos\alpha_3}{h_3^2}\hat{\mathbf{m}}_3^T - \frac{\hat{\mathbf{m}}_{01}^T}{\|\mathbf{e}_0\|^2}\right) \\
&= \frac{\cos\alpha_3}{h_3^2}\hat{\mathbf{m}}_3\hat{\mathbf{n}}_1^T + \frac{\cos\alpha_3}{h_3^2}\hat{\mathbf{n}}_1\hat{\mathbf{m}}_3^T - \frac{\hat{\mathbf{n}}_1\hat{\mathbf{m}}_{01}^T}{\|\mathbf{e}_0\|^2} \\
&= \frac{\cos\alpha_3}{h_3^2}\left(\hat{\mathbf{m}}_3\hat{\mathbf{n}}_1^T + \hat{\mathbf{n}}_1\hat{\mathbf{m}}_3^T\right) - \frac{\hat{\mathbf{n}}_1\hat{\mathbf{m}}_{01}^T}{\|\mathbf{e}_0\|^2} \\
\nabla_{\mathbf{x}_1}\left(\frac{\cos\alpha_3}{h_3}\hat{\mathbf{n}}_1\right) &= \frac{\cos\alpha_3}{h_3}\frac{\hat{\mathbf{m}}_1}{h_1}\hat{\mathbf{n}}_1^T + \hat{\mathbf{n}}_1\left(\frac{\cos\alpha_1}{h_1 h_3}\hat{\mathbf{m}}_3^T + \frac{\hat{\mathbf{m}}_{01}^T}{\|\mathbf{e}_0\|^2}\right) \\
&= \frac{\cos\alpha_3}{h_3 h_1}\hat{\mathbf{m}}_1\hat{\mathbf{n}}_1^T + \frac{\cos\alpha_1}{h_1 h_3}\hat{\mathbf{n}}_1\hat{\mathbf{m}}_3^T + \frac{\hat{\mathbf{n}}_1\hat{\mathbf{m}}_{01}^T}{\|\mathbf{e}_0\|^2} \\
&= \frac{1}{h_1 h_3}\left(\cos\alpha_3\hat{\mathbf{m}}_1\hat{\mathbf{n}}_1^T + \cos\alpha_1\hat{\mathbf{n}}_1\hat{\mathbf{m}}_3^T\right) + \frac{\hat{\mathbf{n}}_1\hat{\mathbf{m}}_{01}^T}{\|\mathbf{e}_0\|^2} \\
\nabla_{\mathbf{x}_2}\left(\frac{\cos\alpha_3}{h_3}\hat{\mathbf{n}}_1\right) &= \frac{\cos\alpha_3}{h_3}\frac{\hat{\mathbf{m}}_{01}}{h_{01}}\hat{\mathbf{n}}_1^T - \hat{\mathbf{n}}_1\frac{\hat{\mathbf{m}}_3^T}{h_{01} h_3} \\
&= \frac{\cos\alpha_3}{h_3 h_{01}}\hat{\mathbf{m}}_{01}\hat{\mathbf{n}}_1^T - \hat{\mathbf{n}}_1\frac{\hat{\mathbf{m}}_3^T}{h_{01} h_3} \\
&= \frac{1}{h_{01} h_3}\left(\cos\alpha_3\hat{\mathbf{m}}_{01}\hat{\mathbf{n}}_1^T - \hat{\mathbf{n}}_1\hat{\mathbf{m}}_3^T\right) \\
\nabla_{\mathbf{x}_3}\left(\frac{\cos\alpha_3}{h_3}\hat{\mathbf{n}}_1\right) &= 0
\end{aligned}$$

These results could also have been obtained by working with the vector expression, but it turns out to be a little more complicated to do so.

Given the above results we can now write down all of the remaining components needed by simple substitution. First we get :

$$\nabla_{\mathbf{x}_0} \left(\frac{\cos \alpha_4}{h_4} \hat{\mathbf{n}}_2 \right) = \frac{\cos \alpha_4}{h_4^2} \left(\hat{\mathbf{m}}_4 \hat{\mathbf{n}}_2^T + \hat{\mathbf{n}}_2 \hat{\mathbf{m}}_4^T \right) - \frac{\hat{\mathbf{n}}_2 \hat{\mathbf{m}}_{02}^T}{\|\mathbf{e}_0\|^2}$$

$$\nabla_{\mathbf{x}_1} \left(\frac{\cos \alpha_4}{h_4} \hat{\mathbf{n}}_2 \right) = \frac{1}{h_2 h_4} \left(\cos \alpha_4 \hat{\mathbf{m}}_2 \hat{\mathbf{n}}_2^T + \cos \alpha_2 \hat{\mathbf{n}}_2 \hat{\mathbf{m}}_4^T \right) + \frac{\hat{\mathbf{n}}_2 \hat{\mathbf{m}}_{02}^T}{\|\mathbf{e}_0\|^2}$$

$$\nabla_{\mathbf{x}_2} \left(\frac{\cos \alpha_4}{h_4} \hat{\mathbf{n}}_2 \right) = 0$$

$$\nabla_{\mathbf{x}_3} \left(\frac{\cos \alpha_4}{h_4} \hat{\mathbf{n}}_2 \right) = \frac{1}{h_{02} h_4} \left(\cos \alpha_4 \hat{\mathbf{m}}_{02} \hat{\mathbf{n}}_2^T - \hat{\mathbf{n}}_2 \hat{\mathbf{m}}_4^T \right)$$

Then, for the second row of the Hessian we get :

$$\nabla_{\mathbf{x}_0} \left(\frac{\cos \alpha_1}{h_1} \hat{\mathbf{n}}_1 \right) = \frac{1}{h_1 h_3} \left(\cos \alpha_1 \hat{\mathbf{m}}_3 \hat{\mathbf{n}}_1^T + \cos \alpha_3 \hat{\mathbf{n}}_1 \hat{\mathbf{m}}_1^T \right) + \frac{\hat{\mathbf{n}}_1 \hat{\mathbf{m}}_{01}^T}{\|\mathbf{e}_0\|^2}$$

$$\nabla_{\mathbf{x}_1} \left(\frac{\cos \alpha_1}{h_1} \hat{\mathbf{n}}_1 \right) = \frac{\cos \alpha_1}{h_1^2} \left(\hat{\mathbf{m}}_1 \hat{\mathbf{n}}_1^T + \hat{\mathbf{n}}_1 \hat{\mathbf{m}}_1^T \right) - \frac{\hat{\mathbf{n}}_1 \hat{\mathbf{m}}_{01}^T}{\|\mathbf{e}_0\|^2}$$

$$\nabla_{\mathbf{x}_2} \left(\frac{\cos \alpha_1}{h_1} \hat{\mathbf{n}}_1 \right) = \frac{1}{h_{01} h_1} \left(\cos \alpha_1 \hat{\mathbf{m}}_{01} \hat{\mathbf{n}}_1^T - \hat{\mathbf{n}}_1 \hat{\mathbf{m}}_1^T \right)$$

$$\nabla_{\mathbf{x}_3} \left(\frac{\cos \alpha_1}{h_1} \hat{\mathbf{n}}_1 \right) = 0$$

and

$$\nabla_{\mathbf{x}_0} \left(\frac{\cos \alpha_2}{h_2} \hat{\mathbf{n}}_2 \right) = \frac{1}{h_2 h_4} \left(\cos \alpha_2 \hat{\mathbf{m}}_4 \hat{\mathbf{n}}_2^T + \cos \alpha_4 \hat{\mathbf{n}}_2 \hat{\mathbf{m}}_2^T \right) + \frac{\hat{\mathbf{n}}_2 \hat{\mathbf{m}}_{02}^T}{\|\mathbf{e}_0\|^2}$$

$$\nabla_{\mathbf{x}_1} \left(\frac{\cos \alpha_2}{h_2} \hat{\mathbf{n}}_2 \right) = \frac{\cos \alpha_2}{h_2^2} \left(\hat{\mathbf{m}}_2 \hat{\mathbf{n}}_2^T + \hat{\mathbf{n}}_2 \hat{\mathbf{m}}_2^T \right) - \frac{\hat{\mathbf{n}}_2 \hat{\mathbf{m}}_{02}^T}{\|\mathbf{e}_0\|^2}$$

$$\nabla_{\mathbf{x}_2} \left(\frac{\cos \alpha_2}{h_2} \hat{\mathbf{n}}_2 \right) = 0$$

$$\nabla_{\mathbf{x}_3} \left(\frac{\cos \alpha_2}{h_2} \hat{\mathbf{n}}_2 \right) = \frac{1}{h_{02} h_2} \left(\cos \alpha_2 \hat{\mathbf{m}}_{02} \hat{\mathbf{n}}_2^T - \hat{\mathbf{n}}_2 \hat{\mathbf{m}}_2^T \right)$$

When we put all this together using equation 42 we get the desired elements

of the Hessian :

$$\nabla_{\mathbf{x}_0}(\nabla_{\mathbf{x}_0}\theta) = \frac{\cos\alpha_3}{h_3^2}\left(\hat{\mathbf{m}}_3\hat{\mathbf{n}}_1^T + \hat{\mathbf{n}}_1\hat{\mathbf{m}}_3^T\right) - \frac{\hat{\mathbf{n}}_1\hat{\mathbf{m}}_{01}^T}{\|\mathbf{e}_0\|^2}$$

$$+ \frac{\cos\alpha_4}{h_4^2}\left(\hat{\mathbf{m}}_4\hat{\mathbf{n}}_2^T + \hat{\mathbf{n}}_2\hat{\mathbf{m}}_4^T\right) - \frac{\hat{\mathbf{n}}_2\hat{\mathbf{m}}_{02}^T}{\|\mathbf{e}_0\|^2}$$

$$\nabla_{\mathbf{x}_1}(\nabla_{\mathbf{x}_0}\theta) = \frac{1}{h_1 h_3}\left(\cos\alpha_3\hat{\mathbf{m}}_1\hat{\mathbf{n}}_1^T + \cos\alpha_1\hat{\mathbf{n}}_1\hat{\mathbf{m}}_3^T\right) + \frac{\hat{\mathbf{n}}_1\hat{\mathbf{m}}_{01}^T}{\|\mathbf{e}_0\|^2}$$

$$+ \frac{1}{h_2 h_4}\left(\cos\alpha_4\hat{\mathbf{m}}_2\hat{\mathbf{n}}_2^T + \cos\alpha_2\hat{\mathbf{n}}_2\hat{\mathbf{m}}_4^T\right) + \frac{\hat{\mathbf{n}}_2\hat{\mathbf{m}}_{02}^T}{\|\mathbf{e}_0\|^2}$$

$$\nabla_{\mathbf{x}_2}(\nabla_{\mathbf{x}_0}\theta) = \frac{1}{h_{01}h_3}\left(\cos\alpha_3\hat{\mathbf{m}}_{01}\hat{\mathbf{n}}_1^T - \hat{\mathbf{n}}_1\hat{\mathbf{m}}_3^T\right)$$

$$\nabla_{\mathbf{x}_3}(\nabla_{\mathbf{x}_0}\theta) = \frac{1}{h_{02}h_4}\left(\cos\alpha_4\hat{\mathbf{m}}_{02}\hat{\mathbf{n}}_2^T - \hat{\mathbf{n}}_2\hat{\mathbf{m}}_4^T\right)$$

and

$$\nabla_{\mathbf{x}_0}(\nabla_{\mathbf{x}_1}\theta) = \frac{1}{h_1 h_3}\left(\cos\alpha_1\hat{\mathbf{m}}_3\hat{\mathbf{n}}_1^T + \cos\alpha_3\hat{\mathbf{n}}_1\hat{\mathbf{m}}_1^T\right) + \frac{\hat{\mathbf{n}}_1\hat{\mathbf{m}}_{01}^T}{\|\mathbf{e}_0\|^2}$$

$$+ \frac{1}{h_2 h_4}\left(\cos\alpha_2\hat{\mathbf{m}}_4\hat{\mathbf{n}}_2^T + \cos\alpha_4\hat{\mathbf{n}}_2\hat{\mathbf{m}}_2^T\right) + \frac{\hat{\mathbf{n}}_2\hat{\mathbf{m}}_{02}^T}{\|\mathbf{e}_0\|^2}$$

$$\nabla_{\mathbf{x}_1}(\nabla_{\mathbf{x}_1}\theta) = \frac{\cos\alpha_1}{h_1^2}\left(\hat{\mathbf{m}}_1\hat{\mathbf{n}}_1^T + \hat{\mathbf{n}}_1\hat{\mathbf{m}}_1^T\right) - \frac{\hat{\mathbf{n}}_1\hat{\mathbf{m}}_{01}^T}{\|\mathbf{e}_0\|^2}$$

$$+ \frac{\cos\alpha_2}{h_2^2}\left(\hat{\mathbf{m}}_2\hat{\mathbf{n}}_2^T + \hat{\mathbf{n}}_2\hat{\mathbf{m}}_2^T\right) - \frac{\hat{\mathbf{n}}_2\hat{\mathbf{m}}_{02}^T}{\|\mathbf{e}_0\|^2}$$

$$\nabla_{\mathbf{x}_2}(\nabla_{\mathbf{x}_1}\theta) = \frac{1}{h_{01}h_1}\left(\cos\alpha_1\hat{\mathbf{m}}_{01}\hat{\mathbf{n}}_1^T - \hat{\mathbf{n}}_1\hat{\mathbf{m}}_1^T\right)$$

$$\nabla_{\mathbf{x}_3}(\nabla_{\mathbf{x}_1}\theta) = \frac{1}{h_{02}h_2}\left(\cos\alpha_2\hat{\mathbf{m}}_{02}\hat{\mathbf{n}}_2^T - \hat{\mathbf{n}}_2\hat{\mathbf{m}}_2^T\right)$$

By comparison with the results in equations 34-41 we see that $\nabla_{\mathbf{x}_i}(\nabla_{\mathbf{x}_j}\theta)^T = (\nabla_{\mathbf{x}_j}(\nabla_{\mathbf{x}_i}\theta)^T)^T$ as expected for $i \neq j$. What remains to be shown is that $\nabla_{\mathbf{x}_i}(\nabla_{\mathbf{x}_i}\theta)^T$ is symmetric for $i \in \{0, 1\}$. To show this we have to show that $\hat{\mathbf{n}}_1\hat{\mathbf{m}}_{01}^T + \hat{\mathbf{n}}_2\hat{\mathbf{m}}_{02}^T$ is symmetric.

Let $\mathbf{n} = \hat{\mathbf{n}}_1 + \hat{\mathbf{n}}_2$ be a vector proportional to the average of the two triangle normals, and let $\mathbf{t} = \mathbf{e}_0 \times \mathbf{n}$ be orthogonal to both $\mathbf{n}$ and $\mathbf{e}_0$ as shown in figure 5. The set $(\hat{\mathbf{e}}_0, \hat{\mathbf{n}}, \hat{\mathbf{t}})$ then constitutes an orthonormal basis for $\mathbb{R}^3$. Consider now a small variation $\delta\mathbf{x}$, and decompose it according to this basis :

$$\delta\mathbf{x} = \delta\mathbf{x}_{\mathbf{e}_0} + \delta\mathbf{x}_{\mathbf{n}} + \delta\mathbf{x}_{\mathbf{t}}$$

Since $\mathbf{e}_0$ is orthogonal to both $\mathbf{m}_{01}$ and $\mathbf{m}_{02}$ it follows that :

$$(\hat{\mathbf{n}}_1\hat{\mathbf{m}}_{01}^T + \hat{\mathbf{n}}_2\hat{\mathbf{m}}_{02}^T)\delta\mathbf{x} = (\hat{\mathbf{n}}_1\hat{\mathbf{m}}_{01}^T + \hat{\mathbf{n}}_2\hat{\mathbf{m}}_{02}^T)(\delta\mathbf{x}_{\mathbf{n}} + \delta\mathbf{x}_{\mathbf{t}})$$
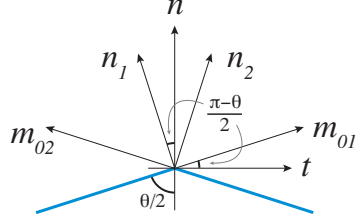
Figure 5. *The cross-section of a hinge.*

To further simplify this, we compute the following quantities with the help of figure 5 :

$$\hat{\mathbf{m}}_{01}^T \delta\mathbf{x_n} = \|\delta\mathbf{x_n}\| \cos\left(\tfrac{\pi}{2} - \tfrac{\theta}{2}\right) = \|\delta\mathbf{x_n}\| \sin\tfrac{\theta}{2}$$
$$\hat{\mathbf{m}}_{02}^T \delta\mathbf{x_n} = \|\delta\mathbf{x_n}\| \cos\left(\tfrac{\pi}{2} - \tfrac{\theta}{2}\right) = \|\delta\mathbf{x_n}\| \sin\tfrac{\theta}{2}$$
$$\hat{\mathbf{m}}_{01}^T \delta\mathbf{x_t} = \|\delta\mathbf{x_t}\| \cos\tfrac{\theta}{2}$$
$$\hat{\mathbf{m}}_{02}^T \delta\mathbf{x_t} = -\|\delta\mathbf{x_t}\| \cos\tfrac{\theta}{2}$$

Using this we get

$$(\hat{\mathbf{n}}_1\hat{\mathbf{m}}_{01}^T + \hat{\mathbf{n}}_2\hat{\mathbf{m}}_{02}^T)\delta\mathbf{x} = \left(\|\delta\mathbf{x_n}\| \sin\tfrac{\theta}{2} + \|\delta\mathbf{x_t}\| \cos\tfrac{\theta}{2}\right)\hat{\mathbf{n}}_1 + \\ \left(\|\delta\mathbf{x_n}\| \sin\tfrac{\theta}{2} - \|\delta\mathbf{x_t}\| \cos\tfrac{\theta}{2}\right)\hat{\mathbf{n}}_2$$

Since

$$\hat{\mathbf{n}}_1 = \hat{\mathbf{n}} \cos\tfrac{\theta}{2} - \hat{\mathbf{t}} \sin\tfrac{\theta}{2}$$
$$\hat{\mathbf{n}}_2 = \hat{\mathbf{n}} \cos\tfrac{\theta}{2} + \hat{\mathbf{t}} \sin\tfrac{\theta}{2}$$

it follows that

$$(\hat{\mathbf{n}}_1\hat{\mathbf{m}}_{01}^T + \hat{\mathbf{n}}_2\hat{\mathbf{m}}_{02}^T)\delta\mathbf{x} = 2\|\delta\mathbf{x_n}\|\hat{\mathbf{n}} \cos\tfrac{\theta}{2} \sin\tfrac{\theta}{2} - 2\|\delta\mathbf{x_t}\|\hat{\mathbf{t}} \sin\tfrac{\theta}{2} \cos\tfrac{\theta}{2}$$
$$= \|\delta\mathbf{x_n}\|\hat{\mathbf{n}} \sin\theta - \|\delta\mathbf{x_t}\|\hat{\mathbf{t}} \sin\theta$$
$$= \sin\theta \left((\delta\mathbf{x} \cdot \hat{\mathbf{n}})\hat{\mathbf{n}} - (\delta\mathbf{x} \cdot \hat{\mathbf{t}})\hat{\mathbf{t}}\right)$$
$$= \sin\theta \left(\hat{\mathbf{n}}\hat{\mathbf{n}}^T - \hat{\mathbf{t}}\hat{\mathbf{t}}^T\right)\delta\mathbf{x}$$

From this we see that $\hat{\mathbf{n}}_1\hat{\mathbf{m}}_{01}^T + \hat{\mathbf{n}}_2\hat{\mathbf{m}}_{02}^T$ must be symmetric, because $\hat{\mathbf{n}}\hat{\mathbf{n}}^T - \hat{\mathbf{t}}\hat{\mathbf{t}}^T$ is the difference between two symmetric matrices, which therefore must be symmetric.

To summarize the result in matrix form, let us define the following short-

hand notations :

$$H_{ij} = \nabla_{\mathbf{x}_j}(\nabla_{\mathbf{x}_i}\theta)^T \tag{51}$$

$$M_{ijk} = \frac{\cos\alpha_i}{h_i h_j}\hat{\mathbf{m}}_j\hat{\mathbf{n}}_k^T \tag{52}$$

$$N_{ij} = \frac{1}{h_{0i}h_j}\hat{\mathbf{n}}_i\hat{\mathbf{m}}_j^T \tag{53}$$

$$S(A) = A + A^T \tag{54}$$

$$B_i = \frac{1}{\|\mathbf{e}_0\|^2}\hat{\mathbf{n}}_i\hat{\mathbf{m}}_{0i}^T \tag{55}$$

Using this we get the following where all terms pertaining to triangle 1 are on the left while all terms pertaining to triangle 2 are on the right :

$$
\begin{aligned}
H_{00} &= S(M_{331}) - B_1 && +S(M_{442}) - B_2 \\
H_{01} &= M_{311} + M_{131}^T + B_1 && +M_{422} + M_{242}^T + B_2 \\
H_{02} &= M_{3.01.1} - N_{13} && \\
H_{03} &= && M_{4.02.2} - N_{24} \\
\\
H_{11} &= S(M_{111}) - B_1 && +S(M_{222}) - B_2 \\
H_{12} &= M_{1.01.1} - N_{11} && \\
H_{13} &= && M_{2.02.2} - N_{22} \\
\\
H_{22} &= -S(N_{1.01}) && \\
H_{23} &= 0 && \\
\\
H_{33} &= && -S(N_{2.02})
\end{aligned}
$$

The remaining blocks are obtained by symmetry. The break-up of the terms into two groups is important, because it shows that we can compute all the contributions from one triangle without knowing anything about the other and vice versa. In fact we can consider each hinge as consisting of two half-hinges where each half-hinge is treated almost identically. The one caveat is that when you change the viewpoint, the implied direction of edge $\mathbf{e}_0$ changes. By carefully going through and writing out all the terms when the view point is flipped around, it turns out that the only place where this has any consequences is the computation of H_{01}. Here, changing the view point will cause B_1 and B_2 to be transposed. Since $B_1 + B_2$ is symmetric, transposing both of them is ok, but if we want to break the computation into two parts then we have to choose a consistent orientation. It should also be noted that while

the complete Hessian is symmetric, the parts associated with each triangle are not since neither B_1 nor B_2 by itself is symmetric.

In order to allow us to break the computation into two parts we will introduce a "conditional transpose" operator denoted by a dagger. With this notation, $B_i^\dagger$ means that B_i should be transposed if the orientation of $\mathbf{e}_0$ in the current view does not agree with a globally chosen orientation of the edge. It doesn't matter what that global orientation is as long as it stays consistent throughout the computations.

6.1.   Assembling the Hessian for a mesh

In this section we combine all the previous results to compute the bending forces and gradients for an entire mesh. As we have seen in the previous section, the Hessian for the bend-angle can be computed by considering one triangle at a time. However, as we've seen in equation 12, the Hessian for the bending energy also contains a term based on the outer product of the gradient of the bend angle with itself. This term is more effectively computed on a per-edge basis, so we will split the overall computation for the force gradients into two stages. One for $\zeta(\theta)H(\theta)$ and one for $\xi(\theta)(\nabla\theta)^T\nabla\theta$. The computation of the forces themselves simply requires $\zeta(\theta)\nabla\theta$.

In order to assemble the forces and the first term of the force gradients for an entire mesh, we have to add the contributions from all the different hinges in the mesh. To see how best to do this, it shall prove advantageous to use a different labeling scheme for vertices, edges, angles, and altitudes. This new scheme is shown in figure 6. All entities are labeled in counter-clockwise order, and edges (and all related quantities) are labeled the same as the opposing vertices. To minimize the potential confusion, the angles in this schemes are referred to as β (as opposed to α), the altitudes as η (as opposed to h), and the inplane edge-normals as $\mathbf{t}$ (as opposed to $\mathbf{m}$). In equations we will refer to the various quantities using an index i, which can take on values $0, 1$ and 2. Arithmetic on all indices is performed modulo 3.

To translate our current results into this new scheme, we first write the contributions to the Hessian for a single half-hinge from triangle 1 in full
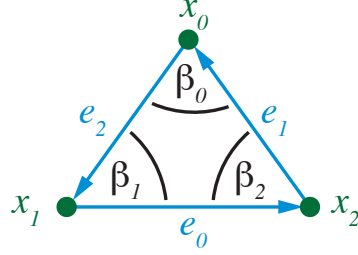
Figure 6. *The triangle centric labeling scheme used to assemble the Hessian for the entire mesh.*

form :

$$H_{00}^* = \frac{\cos \alpha_3}{h_3^2} \left(\hat{\mathbf{m}}_3 \hat{\mathbf{n}}_1^T + \hat{\mathbf{n}}_1 \hat{\mathbf{m}}_3^T \right) - \frac{\hat{\mathbf{n}}_1 \hat{\mathbf{m}}_{01}^T}{\|\mathbf{e}_0\|^2}$$

$$H_{01}^* = \frac{1}{h_1 h_3} \left(\cos \alpha_3 \hat{\mathbf{m}}_1 \hat{\mathbf{n}}_1^T + \cos \alpha_1 \hat{\mathbf{n}}_1 \hat{\mathbf{m}}_3^T \right) + \left(\frac{\hat{\mathbf{n}}_1 \hat{\mathbf{m}}_{01}^T}{\|\mathbf{e}_0\|^2} \right)^{\dagger}$$

$$H_{02}^* = \frac{1}{h_{01} h_3} \left(\cos \alpha_3 \hat{\mathbf{m}}_{01} \hat{\mathbf{n}}_1^T - \hat{\mathbf{n}}_1 \hat{\mathbf{m}}_3^T \right)$$

$$H_{03}^* = 0$$

$$H_{11}^* = \frac{\cos \alpha_1}{h_1^2} \left(\hat{\mathbf{m}}_1 \hat{\mathbf{n}}_1^T + \hat{\mathbf{n}}_1 \hat{\mathbf{m}}_1^T \right) - \frac{\hat{\mathbf{n}}_1 \hat{\mathbf{m}}_{01}^T}{\|\mathbf{e}_0\|^2}$$

$$H_{12}^* = \frac{1}{h_{01} h_1} \left(\cos \alpha_1 \hat{\mathbf{m}}_{01} \hat{\mathbf{n}}_1^T - \hat{\mathbf{n}}_1 \hat{\mathbf{m}}_1^T \right)$$

$$H_{13}^* = 0$$

$$H_{22}^* = - \frac{1}{h_{01}^2} \left(\hat{\mathbf{m}}_{01} \hat{\mathbf{n}}_1^T + \hat{\mathbf{n}}_1 \hat{\mathbf{m}}_{01}^T \right)$$

$$H_{23}^* = 0$$

$$H_{33}^* = 0$$

The asterisk is here used to denote that it's not the full Hessian. Since all the subblocks involving vertex 3 are zero we will drop H_{03}^*, H_{13}^*, H_{23}^*, and H_{33}^* in the following. However, it should be noted that while there are no contributions from the blocks involving vertex 3, there are non-zero contributions for *all* the remaining vertices. As we proceed to assemble the global Hessian for an entire mesh this means that each subblock will receive as many

contributions from a given triangle as there are hinges in that triangle.

The change to the new labels can now be performed by careful but simple substitution. To make it easy to compute the Hessian for all the three half-hinges within a triangle, we will write the result for the new labels relative to an arbitrary index i. The original indices can be recovered by setting $i = 0$ and identifying the indices in figure 6 with those in figure 1 and 2. Finally, we use the abbreviation $l_i = \|\mathbf{e}_i\|$.

$$H^*_{i+1,i+1}(\theta_i) = \frac{\cos\beta_{i-1}}{\eta_{i+1}^2} \left(\hat{\mathbf{t}}_{i+1}\hat{\mathbf{n}}^T + \hat{\mathbf{n}}\hat{\mathbf{t}}_{i+1}^T \right) - \frac{\hat{\mathbf{n}}\hat{\mathbf{t}}_i^T}{l_i^2} \tag{56}$$

$$H^*_{i+1,i-1}(\theta_i) = \frac{1}{\eta_{i-1}\eta_{i+1}} \left(\cos\beta_{i-1}\hat{\mathbf{t}}_{i-1}\hat{\mathbf{n}}^T + \cos\beta_{i+1}\hat{\mathbf{n}}\hat{\mathbf{t}}_{i+1}^T \right) + \left(\frac{\hat{\mathbf{n}}\hat{\mathbf{t}}_i^T}{l_i^2} \right)^{\dagger} \tag{57}$$

$$H^*_{i+1,i}(\theta_i) = \frac{1}{\eta_i\eta_{i+1}} \left(\cos\beta_{i-1}\hat{\mathbf{t}}_i\hat{\mathbf{n}}^T - \hat{\mathbf{n}}\hat{\mathbf{t}}_{i+1}^T \right) \tag{58}$$

$$H^*_{i-1,i-1}(\theta_i) = \frac{\cos\beta_{i+1}}{\eta_{i-1}^2} \left(\hat{\mathbf{t}}_{i-1}\hat{\mathbf{n}}^T + \hat{\mathbf{n}}\hat{\mathbf{t}}_{i-1}^T \right) - \frac{\hat{\mathbf{n}}\hat{\mathbf{t}}_i^T}{l_i^2} \tag{59}$$

$$H^*_{i-1,i}(\theta_i) = \frac{1}{\eta_i\eta_{i-1}} \left(\cos\beta_{i+1}\hat{\mathbf{t}}_i\hat{\mathbf{n}}^T - \hat{\mathbf{n}}\hat{\mathbf{t}}_{i-1}^T \right) \tag{60}$$

$$H^*_{i,i}(\theta_i) = -\frac{1}{\eta_i^2} \left(\hat{\mathbf{t}}_i\hat{\mathbf{n}}^T + \hat{\mathbf{n}}\hat{\mathbf{t}}_i^T \right) \tag{61}$$

Let $H^{\triangle}$ denote the combined Hessian for a triangle. To handle boundary cases correctly we introduce an indicator function σ_i which is 1 if edge i is interior, and 0 if edge i is on the boundary. Given this definition, let $\zeta_i = \sigma_i\zeta(\theta_i)$. We then get the following contribution to $H^{\triangle}$ from the 00 subblock :

$$\begin{aligned} H^{\triangle}_{00} &= \zeta_0 H^*_{00}(\theta_0) + \zeta_1 H^*_{00}(\theta_1) + \zeta_2 H^*_{00}(\theta_2) \\ &= \zeta_0 H^*_{i,i|i=0} + \zeta_1 H^*_{i-1,i-1|i=1} + \zeta_2 H^*_{i+1,i+1|i=2} \end{aligned}$$

Similarly we can write the contributions for all the subblocks :

$$H_{00}^{\triangle} = \zeta_0 H_{i,i|i=0}^* \qquad + \zeta_1 H_{i-1,i-1|i=1}^* \qquad + \zeta_2 H_{i+1,i+1|i=2}^*$$

$$H_{01}^{\triangle} = \zeta_0 H_{i,i+1|i=0}^* \qquad + \zeta_1 H_{i-1,i|i=1}^* \qquad + \zeta_2 H_{i+1,i-1|i=2}^*$$

$$H_{02}^{\triangle} = \zeta_0 H_{i,i-1|i=0}^* \qquad + \zeta_1 H_{i-1,i+1|i=1}^* \qquad + \zeta_2 H_{i+1,i|i=2}^*$$

$$H_{11}^{\triangle} = \zeta_0 H_{i+1,i+1|i=0}^* \qquad + \zeta_1 H_{i,i|i=1}^* \qquad + \zeta_2 H_{i-1,i-1|i=2}^*$$

$$H_{12}^{\triangle} = \zeta_0 H_{i+1,i-1|i=0}^* \qquad + \zeta_1 H_{i,i+1|i=1}^* \qquad + \zeta_2 H_{i-1,i|i=2}^*$$

$$H_{22}^{\triangle} = \zeta_0 H_{i-1,i-1|i=0}^* \qquad + \zeta_1 H_{i+1,i+1|i=1}^* \qquad + \zeta_2 H_{i,i|i=2}^*$$

Using the results from equations 56-61 (transposed if necessary) we can ex-

pand each of these

$$
\begin{aligned}
H_{00}^{\triangle} = {} & -\zeta_0 \frac{1}{\eta_0^2} \left(\hat{\mathbf{t}}_0 \hat{\mathbf{n}}^T + \hat{\mathbf{n}} \hat{\mathbf{t}}_0^T \right) \\
& + \zeta_1 \frac{\cos \beta_2}{\eta_0^2} \left(\hat{\mathbf{t}}_0 \hat{\mathbf{n}}^T + \hat{\mathbf{n}} \hat{\mathbf{t}}_0^T \right) - \zeta_1 \frac{\hat{\mathbf{n}} \hat{\mathbf{t}}_1^T}{l_1^2} \\
& + \zeta_2 \frac{\cos \beta_1}{\eta_0^2} \left(\hat{\mathbf{t}}_0 \hat{\mathbf{n}}^T + \hat{\mathbf{n}} \hat{\mathbf{t}}_0^T \right) - \zeta_2 \frac{\hat{\mathbf{n}} \hat{\mathbf{t}}_2^T}{l_2^2} \\
= {} & \frac{1}{\eta_0^2} (\zeta_1 \cos \beta_2 + \zeta_2 \cos \beta_1 - \zeta_0) \left(\hat{\mathbf{t}}_0 \hat{\mathbf{n}}^T + \hat{\mathbf{n}} \hat{\mathbf{t}}_0^T \right) - \zeta_1 \frac{\hat{\mathbf{n}} \hat{\mathbf{t}}_1^T}{l_1^2} - \zeta_2 \frac{\hat{\mathbf{n}} \hat{\mathbf{t}}_2^T}{l_2^2} \\
H_{01}^{\triangle} = {} & \zeta_0 \frac{1}{\eta_0 \eta_1} \left(\cos \beta_2 \hat{\mathbf{n}} \hat{\mathbf{t}}_0^T - \hat{\mathbf{t}}_1 \hat{\mathbf{n}}^T \right) \\
& + \zeta_1 \frac{1}{\eta_1 \eta_0} \left(\cos \beta_2 \hat{\mathbf{t}}_1 \hat{\mathbf{n}}^T - \hat{\mathbf{n}} \hat{\mathbf{t}}_0^T \right) \\
& + \zeta_2 \frac{1}{\eta_1 \eta_0} \left(\cos \beta_1 \hat{\mathbf{t}}_1 \hat{\mathbf{n}}^T + \cos \beta_0 \hat{\mathbf{n}} \hat{\mathbf{t}}_0^T \right) + \zeta_2 \left(\frac{\hat{\mathbf{n}} \hat{\mathbf{t}}_2^T}{l_2^2} \right)^{\dagger} \\
= {} & \frac{1}{\eta_0 \eta_1} \left((\zeta_2 \cos \beta_0 + \zeta_0 \cos \beta_2 - \zeta_1) \hat{\mathbf{n}} \hat{\mathbf{t}}_0^T + (\zeta_2 \cos \beta_1 + \zeta_1 \cos \beta_2 - \zeta_0) \hat{\mathbf{t}}_1 \hat{\mathbf{n}}^T \right) + \zeta_2 \frac{\hat{\mathbf{n}} \hat{\mathbf{t}}_2^T}{l_2^2} \\
H_{02}^{\triangle} = {} & \zeta_0 \frac{1}{\eta_0 \eta_2} \left(\cos \beta_1 \hat{\mathbf{n}} \hat{\mathbf{t}}_0^T - \hat{\mathbf{t}}_2 \hat{\mathbf{n}}^T \right) \\
& + \zeta_1 \frac{1}{\eta_0 \eta_2} \left(\cos \beta_0 \hat{\mathbf{n}} \hat{\mathbf{t}}_0^T + \cos \beta_2 \hat{\mathbf{t}}_2 \hat{\mathbf{n}}^T \right) + \zeta_1 \left(\frac{\hat{\mathbf{t}}_1 \hat{\mathbf{n}}^T}{l_1^2} \right)^{\dagger} \\
& + \zeta_2 \frac{1}{\eta_2 \eta_0} \left(\cos \beta_1 \hat{\mathbf{t}}_2 \hat{\mathbf{n}}^T - \hat{\mathbf{n}} \hat{\mathbf{t}}_0^T \right) \\
= {} & \frac{1}{\eta_0 \eta_2} \left((\zeta_1 \cos \beta_0 + \zeta_0 \cos \beta_1 - \zeta_2) \hat{\mathbf{n}} \hat{\mathbf{t}}_0^T + (\zeta_2 \cos \beta_1 + \zeta_1 \cos \beta_2 - \zeta_0) \hat{\mathbf{t}}_2 \hat{\mathbf{n}}^T \right) + \zeta_1 \frac{\hat{\mathbf{t}}_1 \hat{\mathbf{n}}^T}{l_1^2} \\
H_{11}^{\triangle} = {} & \zeta_0 \frac{\cos \beta_2}{\eta_1^2} \left(\hat{\mathbf{t}}_1 \hat{\mathbf{n}}^T + \hat{\mathbf{n}} \hat{\mathbf{t}}_1^T \right) - \zeta_0 \frac{\hat{\mathbf{n}} \hat{\mathbf{t}}_0^T}{l_0^2} \\
& - \zeta_1 \frac{1}{\eta_1^2} \left(\hat{\mathbf{t}}_1 \hat{\mathbf{n}}^T + \hat{\mathbf{n}} \hat{\mathbf{t}}_1^T \right) \\
& + \zeta_2 \frac{\cos \beta_0}{\eta_1^2} \left(\hat{\mathbf{t}}_1 \hat{\mathbf{n}}^T + \hat{\mathbf{n}} \hat{\mathbf{t}}_1^T \right) - \zeta_2 \frac{\hat{\mathbf{n}} \hat{\mathbf{t}}_2^T}{l_2^2} \\
= {} & \frac{1}{\eta_1^2} (\zeta_2 \cos \beta_0 + \zeta_0 \cos \beta_2 - \zeta_1) \left(\hat{\mathbf{t}}_1 \hat{\mathbf{n}}^T + \hat{\mathbf{n}} \hat{\mathbf{t}}_1^T \right) - \zeta_0 \frac{\hat{\mathbf{n}} \hat{\mathbf{t}}_0^T}{l_0^2} - \zeta_2 \frac{\hat{\mathbf{n}} \hat{\mathbf{t}}_2^T}{l_2^2}
\end{aligned}
$$

$$H_{12}^{\triangle} = \zeta_0 \frac{1}{\eta_2 \eta_1} \left(\cos \beta_2 \hat{\mathbf{t}}_2 \hat{\mathbf{n}}^T + \cos \beta_1 \hat{\mathbf{n}} \hat{\mathbf{t}}_1^T \right) + \zeta_0 \left(\frac{\hat{\mathbf{n}} \hat{\mathbf{t}}_0^T}{l_0^2} \right)^{\dagger}$$

$$+ \zeta_1 \frac{1}{\eta_1 \eta_2} \left(\cos \beta_0 \hat{\mathbf{n}} \hat{\mathbf{t}}_1^T - \hat{\mathbf{t}}_2 \hat{\mathbf{n}}^T \right)$$

$$+ \zeta_2 \frac{1}{\eta_2 \eta_1} \left(\cos \beta_0 \hat{\mathbf{t}}_2 \hat{\mathbf{n}}^T - \hat{\mathbf{n}} \hat{\mathbf{t}}_1^T \right)$$

$$= \frac{1}{\eta_1 \eta_2} \left((\zeta_2 \cos \beta_0 + \zeta_0 \cos \beta_2 - \zeta_1) \hat{\mathbf{t}}_2 \hat{\mathbf{n}}^T + (\zeta_1 \cos \beta_0 + \zeta_0 \cos \beta_1 - \zeta_2) \hat{\mathbf{n}} \hat{\mathbf{t}}_1^T \right) + \zeta_0 \frac{\hat{\mathbf{n}} \hat{\mathbf{t}}_0^T}{l_0^2}$$

$$H_{22}^{\triangle} = \zeta_0 \frac{\cos \beta_1}{\eta_2^2} \left(\hat{\mathbf{t}}_2 \hat{\mathbf{n}}^T + \hat{\mathbf{n}} \hat{\mathbf{t}}_2^T \right) - \zeta_0 \frac{\hat{\mathbf{n}} \hat{\mathbf{t}}_0^T}{l_0^2}$$

$$+ \zeta_1 \frac{\cos \beta_0}{\eta_2^2} \left(\hat{\mathbf{t}}_2 \hat{\mathbf{n}}^T + \hat{\mathbf{n}} \hat{\mathbf{t}}_2^T \right) - \zeta_1 \frac{\hat{\mathbf{n}} \hat{\mathbf{t}}_1^T}{l_1^2}$$

$$- \zeta_2 \frac{1}{\eta_2^2} \left(\hat{\mathbf{t}}_2 \hat{\mathbf{n}}^T + \hat{\mathbf{n}} \hat{\mathbf{t}}_2^T \right)$$

$$= \frac{1}{\eta_2^2} (\zeta_1 \cos \beta_0 + \zeta_0 \cos \beta_1 - \zeta_2) \left(\hat{\mathbf{t}}_2 \hat{\mathbf{n}}^T + \hat{\mathbf{n}} \hat{\mathbf{t}}_2^T \right) - \zeta_0 \frac{\hat{\mathbf{n}} \hat{\mathbf{t}}_0^T}{l_0^2} - \zeta_1 \frac{\hat{\mathbf{n}} \hat{\mathbf{t}}_1^T}{l_1^2}$$

Clearly, the numbering of the vertices is arbitrary, so in the end we should get the same Hessian for any cyclic permutation of the indices. As a sanity check we should therefore see the following relationship :

$$H_{00} \to H_{11} \to H_{22}$$

and

$$H_{01} \to H_{12} \to H_{20} = H_{02}^T$$

It's important to notice the transpose at the end of this last sequence since it is essential for the results to match. To verify the results, we will rewrite

them in an easy to compare fashion :

$$H_{00}^{\triangle} = \frac{1}{\eta_0^2}(\zeta_2 \cos\beta_1 + \zeta_1 \cos\beta_2 - \zeta_0)\left(\hat{\mathbf{t}}_0\hat{\mathbf{n}}^T + \hat{\mathbf{n}}\hat{\mathbf{t}}_0^T\right) - \zeta_1 \frac{\hat{\mathbf{n}}\hat{\mathbf{t}}_1^T}{l_1^2} - \zeta_2 \frac{\hat{\mathbf{n}}\hat{\mathbf{t}}_2^T}{l_2^2}$$

$$H_{11}^{\triangle} = \frac{1}{\eta_1^2}(\zeta_0 \cos\beta_2 + \zeta_2 \cos\beta_0 - \zeta_1)\left(\hat{\mathbf{t}}_1\hat{\mathbf{n}}^T + \hat{\mathbf{n}}\hat{\mathbf{t}}_1^T\right) - \zeta_0 \frac{\hat{\mathbf{n}}\hat{\mathbf{t}}_0^T}{l_0^2} - \zeta_2 \frac{\hat{\mathbf{n}}\hat{\mathbf{t}}_2^T}{l_2^2}$$

$$H_{22}^{\triangle} = \frac{1}{\eta_2^2}(\zeta_1 \cos\beta_0 + \zeta_0 \cos\beta_1 - \zeta_2)\left(\hat{\mathbf{t}}_2\hat{\mathbf{n}}^T + \hat{\mathbf{n}}\hat{\mathbf{t}}_2^T\right) - \zeta_0 \frac{\hat{\mathbf{n}}\hat{\mathbf{t}}_0^T}{l_0^2} - \zeta_1 \frac{\hat{\mathbf{n}}\hat{\mathbf{t}}_1^T}{l_1^2}$$

$$H_{01}^{\triangle} = \frac{1}{\eta_0\eta_1}\left((\zeta_2 \cos\beta_1 + \zeta_1 \cos\beta_2 - \zeta_0)\hat{\mathbf{t}}_1\hat{\mathbf{n}}^T + (\zeta_2 \cos\beta_0 + \zeta_0 \cos\beta_2 - \zeta_1)\hat{\mathbf{n}}\hat{\mathbf{t}}_0^T\right) + \zeta_2\left(\frac{\hat{\mathbf{n}}\hat{\mathbf{t}}_2^T}{l_2^2}\right)^{\dagger}$$

$$H_{12}^{\triangle} = \frac{1}{\eta_1\eta_2}\left((\zeta_0 \cos\beta_2 + \zeta_2 \cos\beta_0 - \zeta_1)\hat{\mathbf{t}}_2\hat{\mathbf{n}}^T + (\zeta_0 \cos\beta_1 + \zeta_1 \cos\beta_0 - \zeta_2)\hat{\mathbf{n}}\hat{\mathbf{t}}_1^T\right) + \zeta_0\left(\frac{\hat{\mathbf{n}}\hat{\mathbf{t}}_0^T}{l_0^2}\right)^{\dagger}$$

$$(H_{02}^{\triangle})^T = \frac{1}{\eta_0\eta_2}\left((\zeta_1 \cos\beta_0 + \zeta_0 \cos\beta_1 - \zeta_2)\hat{\mathbf{t}}_0\hat{\mathbf{n}}^T + (\zeta_1 \cos\beta_2 + \zeta_2 \cos\beta_1 - \zeta_0)\hat{\mathbf{n}}\hat{\mathbf{t}}_2^T\right) + \zeta_1\left(\frac{\hat{\mathbf{n}}\hat{\mathbf{t}}_1^T}{l_1^2}\right)^{\dagger}$$

Given this formulation it should be obvious that there are significant computational advantages over a naive computation of the Hessian for each hinge. In particular, we see that there are really only three different outer products which have to be computed, and there are also only three different cosine expressions.

Let

$$\gamma_i = \zeta_{i-1} \cos\beta_{i+1} + \zeta_{i+1} \cos\beta_{i-1} - \zeta_i$$
$$T_i = \hat{\mathbf{n}}\hat{\mathbf{t}}_i^T$$

We then get :

$$H_{00}^{\triangle} = \frac{1}{\eta_0^2}\gamma_0\left(T_0^T + T_0\right) - \frac{\zeta_1}{l_1^2}T_1 - \frac{\zeta_2}{l_2^2}T_2$$

$$H_{11}^{\triangle} = \frac{1}{\eta_1^2}\gamma_1\left(T_1^T + T_1\right) - \frac{\zeta_0}{l_0^2}T_0 - \frac{\zeta_2}{l_2^2}T_2$$

$$H_{22}^{\triangle} = \frac{1}{\eta_2^2}\gamma_2\left(T_2^T + T_2\right) - \frac{\zeta_0}{l_0^2}T_0 - \frac{\zeta_1}{l_1^2}T_1$$

$$H_{01}^{\triangle} = \frac{1}{\eta_0\eta_1}\left(\gamma_0 T_1^T + \gamma_1 T_0\right) + \frac{\zeta_2}{l_2^2}T_2^{\dagger}$$

$$H_{12}^{\triangle} = \frac{1}{\eta_1\eta_2}\left(\gamma_1 T_2^T + \gamma_2 T_1\right) + \frac{\zeta_0}{l_0^2}T_0^{\dagger}$$

$$(H_{02}^{\triangle})^T = \frac{1}{\eta_0\eta_2}\left(\gamma_2 T_0^T + \gamma_0 T_2\right) + \frac{\zeta_1}{l_1^2}T_1^{\dagger}$$

Here we also notice that the term $\zeta_i T_i / l_i^2$ is used three times. Taking this into account we need to compute 3 outer products, 12 scaled versions of these outer products and 15 matrix additions of 3×3 matrices in order to compute the lower half of the Hessian for one triangle. The other half follows from symmetry.

In comparison, the Hessian for a single full hinge requires six different outer products, 20 scaled versions of these, and 18 matrix additions. For a regular mesh there are twice as many edges as faces, so the total (relative) cost for the entire mesh becomes 12 outer products, 40 scale operations and 36 additions vs. 3, 12 and 15 operations. Roughly speaking, it should therefore provide a $3x$ speedup. However, the locality of the above computation also helps cache-coherency, which can provide a significant speedup on its own.

By combining the results presented so far we get the following algorithm for computing the first term of the Hessian of the bending energy for the entire mesh :

- For each triangle

    compute face normal, $\mathbf{n}$

    compute triangle area, $2A = \|\mathbf{n}\|$

    normalize face normal, $\hat{\mathbf{n}} = \mathbf{n}/\|\mathbf{n}\|$

- For each edge

    compute edge vector, $\mathbf{e}$

    compute edge length, $l = \|\mathbf{e}\|$

    compute normalized edge vector, $\hat{\mathbf{e}} = \mathbf{e}/\|\mathbf{e}\|$

    compute $\zeta(\theta)$ and $v(\theta)$

- For each triangle and $i \in \{0, 1, 2\}$:

    compute $\cos \beta_i = -\hat{\mathbf{e}}_{i-1} \cdot \hat{\mathbf{e}}_{i+1}$. Store with vertex.

    compute the three altitudes, $\eta_i = 2A/l_i$. Store with vertex.

    compute the three outward-pointing edge-normals, $\hat{\mathbf{t}}_i = \hat{\mathbf{e}}_i \times \hat{\mathbf{n}}$.

    compute the three outer products, $T_i = \hat{\mathbf{n}}\hat{\mathbf{t}}_i^T$

    compute $\tilde{T}_i = \zeta_i T_i / l_i^2$.

    compute $\gamma_i = \zeta_{i-1} \cos \beta_{i+1} + \zeta_{i+1} \cos \beta_{i-1} - \zeta_i$

    set $H_{ii}^{\triangle} = \frac{\gamma_i}{\eta_i^2}(T_i^T + T_i)$

    update $H_{i-1,i-1}^{\triangle} -= \tilde{T}_i$ and $H_{i+1,i+1}^{\triangle} -= \tilde{T}_i$

    set $H_{i-1,i}^{\triangle} = \frac{\gamma_{i-1}}{\eta_{i-1}\eta_i} T_i^T$

update $H^{\triangle}_{i+1,i-1} += \tilde{T}_i^{\dagger}$

update $H^{\triangle}_{i,i+1} += \frac{\gamma_{i+1}}{\eta_i \eta_{i+1}} T_i^T$

set $H^{\triangle}_{10} = (H^{\triangle}_{01})^T$, $H^{\triangle}_{21} = (H^{\triangle}_{12})^T$, and $H^{\triangle}_{02} = (H^{\triangle}_{20})^T$

add $H^{\triangle}$ to the global matrix

In order to finish the computation of the Hessian for the bending energy we need to compute the term $\xi(\theta)(\nabla\theta)^T \nabla\theta$. It can be shown that the *sum* of $\nabla\theta$ for all edges incident to a given vertex almost falls out for free from the above computation. However, this would not provide $\nabla\theta$ for each *edge*. Since $\xi(\theta)(\nabla\theta)^T\nabla\theta$ contains terms which depend on two triangles it is also not practical to do a per-triangle decomposition as above. In the end it is most efficient to compute $\xi(\theta)(\nabla\theta)^T$ and $\nabla\theta$ and then compute the outer product by simple multiplication.

7.   Lack of positive definiteness

In many situations it is desirable for a matrix to be positive definite. In particular this is a requirement for being able to use the conjugate gradient method to solve the corresponding set of linear equations.

In this section we will show that the Hessian of the bending energy is *not* guaranteed to be positive definite. In fact, it is at best positive semi-definite. The fact that it is not always positive definite is not at all surprising. It was shown in by [Coleman and Noll 59] that a strictly convex energy is incompatible with frame invariance which means that a strictly convex energy must depend on the choice of coordinate system. Strict convexity also implies *uniqueness* of all solutions, which precludes phenomena like buckling [Hill 57].

To show that the Hessian is at best positive semi-definite we note that the bending energy is designed to be invariant under rigid body transformations. In particular this means that it is invariant under translations. Hence if we translate all vertices by the same amount, neither the energy nor the forces change. We might therefore guess that 0 is an eigenvalue for the Hessian corresponding to eigenvectors consisting of uniform translations. This is in fact the case which we can prove by considering a uniform translation in configuration space

$$\mathbf{v} = [\mathbf{u}, \mathbf{u}, \mathbf{u}, \mathbf{u}].$$

Here $\mathbf{u}$ is the translation vector in $\mathbb{R}^3$. With this notation we need to show that

$$-H(E_b)\mathbf{v} = \left(\zeta(\theta)H(\theta) + \xi(\theta)\nabla\theta^T\nabla\theta\right)\mathbf{v} = \mathbf{0}$$

To do this, let us first consider $(\nabla\theta)\mathbf{v}$. Given the expressions in equations 18-21 we get

$$(\nabla\theta)\mathbf{v} = \left(\frac{\cos\alpha_3}{h_3} + \frac{\cos\alpha_1}{h_1} - \frac{1}{h_{01}} \right) \hat{\mathbf{n}}_1^T \mathbf{u}$$
$$+ \left(\frac{\cos\alpha_4}{h_4} + \frac{\cos\alpha_2}{h_2} - \frac{1}{h_{02}} \right) \hat{\mathbf{n}}_2^T \mathbf{u}$$

If we consider just the first of these terms, then we see that

$$\frac{\cos\alpha_3}{h_3} + \frac{\cos\alpha_1}{h_1} - \frac{1}{h_{01}} = \frac{\|\mathbf{e}_3\|\cos\alpha_3}{2A_1} + \frac{\|\mathbf{e}_1\|\cos\alpha_1}{2A_1} - \frac{\|\mathbf{e}_0\|}{2A_1}$$
$$= \frac{1}{2A_1} \left(\|\mathbf{e}_3\|\frac{\gamma\|\mathbf{e}_0\|}{\|\mathbf{e}_3\|} + \|\mathbf{e}_1\|\frac{(1-\gamma)\|\mathbf{e}_0\|}{\|\mathbf{e}_1\|} - \|\mathbf{e}_0\| \right)$$
$$= 0 \tag{62}$$

Here, $\gamma\|\mathbf{e}_0\|$ is the distance from $\mathbf{x}_0$ to the projection of $\mathbf{x}_2$ onto $\mathbf{e}_0$, and $(1-\gamma)\|\mathbf{e}_0\|$ is the distance from the same projection to $\mathbf{x}_1$. The fact that the second term is also 0 follows by symmetry, and as a consequence we see that $(\nabla\theta)\mathbf{v} = 0$.

Computing $H(\theta)\mathbf{v}$ is a little more tedious, but essentially straight forward. After collecting terms, the scale-factors we just considered for $(\nabla\theta)\mathbf{v}$ show up again which means that those terms vanish. The remaining terms are scaled by factors of the following form :

$$\frac{\hat{\mathbf{m}}_3}{h_3} + \frac{\hat{\mathbf{m}}_1}{h_1} + \frac{\hat{\mathbf{m}}_{01}}{h_{01}} = \frac{\|\mathbf{e}_3\|\hat{\mathbf{m}}_3}{2A_1} + \frac{\|\mathbf{e}_1\|\hat{\mathbf{m}}_1}{2A_1} + \frac{\|\mathbf{e}_0\|\hat{\mathbf{m}}_{01}}{2A_1}$$
$$= \frac{1}{2A_1} (\mathbf{m}_3 + \mathbf{m}_1 + \mathbf{m}_{01})$$
$$= 0$$

The last equality follows from the fact that $\mathbf{e}_3 + \mathbf{e}_1 + \mathbf{e}_{01} = \mathbf{0}$ since all the edges connect together and $\mathbf{m}_i$ is just an inplane rotation of $\mathbf{e}_i$. From all this it follows that $H(\theta)\mathbf{v} = \mathbf{0}$, so $\mathbf{v}$ is an eigenvector corresponding to a zero eigenvalue. And since $\mathbf{u}$, which is used to define $\mathbf{v}$, can be chosen arbitrarily in $\mathbb{R}^3$ it follows that the zero eigenvalue has at least geometric multiplicity 3.

It should be noted that all of the above computations are valid for any configuration, so the Hessian of the bending energy is *never* strictly positive definite.

In turns out that there is one more zero eigenvalue. This eigenvalue corresponds to uniform scaling along the hinge edge. It is obvious that such a scaling won't change the hinge-angle, so the energy remains unchanged, and

since the eigenvalue is zero, it follows that the forces also don't change. The corresponding eigenvector can be written as :

$$\mathbf{v} = [\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]$$

where $\mathbf{v}_i = (\mathbf{x}_i \cdot \mathbf{e}_0)\mathbf{e}_0$. It is easy to see that $\mathbf{v}$ is orthogonal to $\nabla\theta$ since $\mathbf{e}_0$ is orthogonal to $\mathbf{n}_1$ and $\mathbf{n}_2$. What remains to show is that $H(\theta)\mathbf{v} = 0$. To do this we will use the notation introduced in equation 51-55.

First we note that $\hat{\mathbf{n}}_i$ and $\hat{\mathbf{m}}_{0i}$ are both perpendicular to $\mathbf{e}_0$, so all terms involving $M_{ijk}\mathbf{v}_l$ and $B_i\mathbf{v}_l$ are zero. Similarly all terms involving $N_{ij}^T\mathbf{v}_l$ and $N_{i.0j}\mathbf{v}_l$ are zero. Let H_0 denote the first block row of $H(\theta)$. After collecting terms we get :

$$H_0\mathbf{v} = \frac{1}{h_3}\left(\frac{\cos\alpha_3}{h_3}(\mathbf{x}_0 \cdot \mathbf{e}_0) + \frac{\cos\alpha_1}{h_1}(\mathbf{x}_1 \cdot \mathbf{e}_0) - \frac{1}{h_{01}}(\mathbf{x}_2 \cdot \mathbf{e}_0)\right)\hat{\mathbf{n}}_1\hat{\mathbf{m}}_3^T\mathbf{e}_0 +$$

$$\frac{1}{h_4}\left(\frac{\cos\alpha_4}{h_4}(\mathbf{x}_0 \cdot \mathbf{e}_0) + \frac{\cos\alpha_2}{h_2}(\mathbf{x}_1 \cdot \mathbf{e}_0) - \frac{1}{h_{02}}(\mathbf{x}_3 \cdot \mathbf{e}_0)\right)\hat{\mathbf{n}}_2\hat{\mathbf{m}}_4^T\mathbf{e}_0$$

For the first term we have :

$$\frac{\cos\alpha_3}{h_3}(\mathbf{x}_0 \cdot \mathbf{e}_0) + \frac{\cos\alpha_1}{h_1}(\mathbf{x}_1 \cdot \mathbf{e}_0) - \frac{1}{h_{01}}(\mathbf{x}_2 \cdot \mathbf{e}_0) =$$

$$\frac{\cos\alpha_3}{h_3}(\mathbf{x}_0 \cdot \mathbf{e}_0) + \frac{\cos\alpha_1}{h_1}((\mathbf{x}_0 + \mathbf{e}_0) \cdot \mathbf{e}_0) - \frac{1}{h_{01}}((\mathbf{x}_0 + \mathbf{e}_1) \cdot \mathbf{e}_0) =$$

$$\frac{\cos\alpha_1}{h_1}(\mathbf{e}_0 \cdot \mathbf{e}_0) - \frac{1}{h_{01}}(\mathbf{e}_1 \cdot \mathbf{e}_0) =$$

$$\frac{\cos\alpha_1}{h_1}\|\mathbf{e}_0\|^2 - \frac{\cos\alpha_1}{h_{01}}\|\mathbf{e}_1\|\|\mathbf{e}_0\| =$$

$$\cos\alpha_1\|\mathbf{e}_0\|\left(\frac{\|\mathbf{e}_0\|}{h_1} - \frac{\|\mathbf{e}_1\|}{h_{01}}\right) = 0$$

The second equality here follows from equation 62. The last equality follows from the fact that the two right triangles with sides $(\mathbf{e}_1, h_{01})$ and $(\mathbf{e}_0, h_1)$ are similar (all their angles are the same), so the ratios between the lengths of their sides must be the same. By symmetry it follows that the second term in the expression for $H_0\mathbf{v}$ is also zero, so we have $H_0\mathbf{v} = 0$. The computations for $H_1\mathbf{v}$ follow the same pattern and $H_2\mathbf{v}$ and $H_3\mathbf{v}$ are easily shown to be zero since all terms turn out to be zero. As a result we have $H(\theta)\mathbf{v} = 0$, so $\mathbf{v}$ is an eigenvector corresponding to a zero eigenvalue.

8.   Degeneracies

There are two types of first order degeneracies : Edge collapses and altitude collapses. Both of these will cause one of more of the altitudes in a hinge to

become zero. This will formally lead to division by zero in the computation of both the gradient and the Hessian for the bending energy. For an edge collapse it is easy to show that all the relevant terms have well-defined limit values as the edge length goes to zero, so one could include this as a special case. Unfortunately, an altitude collapse is not as easy to handle.

In theory the tensile forces should prevent the elements from ever degenerating, but in practice it that doesn't quite work. As an example the St. Venant-Kirchhoff constitutive model does not have a singularity for $\det F = 0$ which means that a finite amount of energy can cause a triangle to degenerate. Other constitutive models do have such a singularity, but they are exceedingly ill-conditioned in the near-degenerate configuration.

References

[Audoly and Pomeau 10] Basile Audoly and Yves Pomeau. *Elasticity and Geometry: From hair curls to the nonlinear response of shells*. Oxford University Press, 2010.

[Bergou et al. 10] Miklós Bergou, Basile Audoly, Etienne Vouga, Max Wardetzky, and Eitan Grinspun. "Discrete viscous threads." *ACM Trans. Graph.* 29 (2010), 116:1–116:10.

[Bridson et al. 03] Robert Bridson, Sebastian Marino, and Ronald Fedkiw. "Simulation of clothing with folds and wrinkles." In *SCA '03: Proceedings of the 2003 ACM SIGGRAPH/Eurographics Symposium on Computer Animation*, pp. 28–36. Eurographics Association, 2003.

[Cohen-Steiner and Morvan 03] David Cohen-Steiner and Jean-Marie Morvan. "Restricted Delaunay Triangulations and Normal Cycle." In *SCG '03: Proceedings of the nineteenth annual symposium on Computational geometry*, pp. 312–321. New York, NY, USA: ACM, 2003.

[Coleman and Noll 59] Bernard D. Coleman and Walter Noll. "On the Thermostatics of Continuous Media." *Archive for Rational Mechanics and Analysis* 4:1 (1959), 97–128.

[Gingold et al. 04] Yotam Gingold, Adrian Secord, Jefferson Y. Han, Eitan Grinspun, and Denis Zorin. "A Discrete Model for Inelastic Deformation of Thin Shells." Technical report, Courant Institute of Mathematical Sciences, New York University, 2004.

[Grinspun et al. 03] Eitan Grinspun, Anil N. Hirani, Mathieu Desbrun, and Peter Schröder. "Discrete shells." In *SCA '03: Proceedings of the 2003 ACM*

SIGGRAPH/Eurographics Symposium on Computer Animation, pp. 62–67. Eurographics Association, 2003.

[Hill 57] R. Hill. "On uniqueness and stability in the theory of finite elastic strain." *Journal of the Mechanics and Physics of Solids* 5:4 (1957), 229–241.

[Kharevych et al. 06] L. Kharevych, Weiwei Yang, Y. Tong, E. Kanso, J. E. Marsden, P. Schröder, and M. Desbrun. "Geometric, variational integrators for computer animation." In *Proceedings of the 2006 ACM SIGGRAPH/Eurographics symposium on Computer animation*, *SCA '06*, pp. 43–51. Eurographics Association, 2006.

[Wardetzky et al. 07] Max Wardetzky, Miklós Bergou, David Harmon, Denis Zorin, and Eitan Grinspun. "Discrete Quadratic Curvature Energies." *Computer Aided Geometric Design* 24:8-9 (2007), 499–518.

Web Information: